

WHITTAKER FUNCTIONS ON METAPLECTIC COVERS OF $GL(r)$

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ABSTRACT. This paper establishes a combinatorial link between different approaches to constructing Whittaker functions on a metaplectic group over a non-archimedean local field. We prove a metaplectic analogue of Tokuyama's Theorem and give a crystal description of polynomials related to Iwahori-Whittaker functions. The proof relies on formulas of metaplectic Demazure and Demazure-Lusztig operators, proved previously in joint work with Gautam Chinta and Paul E. Gunnells [CGP14].

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1. INTRODUCTION

1.1. **Motivation.** The study of metaplectic groups was initiated by Matsumoto [Mat69]. Analytic number theory, in particular questions about the mean values of L -functions led to research on multiple Dirichlet series, which in turn motivated interest in Whittaker coefficients of metaplectic Eisenstein series. Whittaker functions are higher dimension generalizations of Bessel functions and are associated to principal series representations of a reductive group over a local field. Kubota [Kub71] was the first to closely examine Eisenstein series on higher covers of GL_2 , and the theory of associated Whittaker functions was further developed by Kazhdan and Patterson [KP84]. In recent years, this development gained further impetus from unexpected connections to other areas, such as combinatorial representation theory, the geometry of Schubert varieties, and solvable lattice models.

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While the theory of metaplectic Whittaker functions is familiar in the case of double covers of reductive groups, it is less well understood in the case of higher covers.

1.1.1. The Casselman-Shalika formula. The Casselman-Shalika formula is an explicit formula for values of a spherical Whittaker function over a local p -adic field in terms of a character of a reductive group. It is a central result in understanding the local and global theory of automorphic forms and their L -functions. A metaplectic analogue, describing Whittaker functions on n -fold covers of a reductive group, has similar significance in the study of Dirichlet series of several variables. Different approaches to generalize the Casselman-Shalika formula to the metaplectic setting have recently emerged.

1.1.2. Metaplectic analogues. Chinta-Offen [CO13] and McNamara [McN16] generalize the Casselman-Shalika formula by replacing the character with a metaplectic analogue: a sum over the Weyl group involving a modified action of the Weyl group that depends on the metaplectic cover. Brubaker-Bump-Friedberg [BBF11a] and McNamara [McN11] express a type A Whittaker function as a sum over a crystal base. Both constructions produce the Whittaker function as a polynomial determined by combinatorial data: the root datum of the group, a dominant weight, and the degree n of the metaplectic cover. The first one handles all types of root datum, while the second one makes it possible to compute the coefficients of the polynomial individually. The fact that the descriptions are purely combinatorial in nature, and rely heavily on Weyl group combinatorics and on the structure of the crystal graph, indicates that deeper properties of these constructions can be understood using methods of combinatorial representation theory.

1.1.3. Combinatorial link. In the present paper we develop a combinatorial understanding of the relationship of the two approaches described in section 1.1.2. This is one aspect of our main result (Theorem 1); an other is giving a crystal description of polynomials related to Iwahori-Whittaker functions. Both of these aspects will be made explicit in section 9.

Furthermore, both approaches to constructing Whittaker functions, i.e. summing over the Weyl group, or, respectively, over a crystal graph, also make sense in the nonmetaplectic setting. In this special case, a theorem of Tokuyama provides a combinatorial link between them [Tok88]. In the metaplectic setting, the constructions of section 1.1.2 naturally extend the meaning of respective sides of Tokuyama’s identity. Thus explicitly relating the two constructions is in essence proving a metaplectic analogue of Tokuyama’s formula. Viewed purely as an identity about the combinatorial data, the special case of Theorem 1 stated as Theorem 2 is this metaplectic analogue.

1.2. Methods and tools. The connection between Tokuyama’s theorem and the constructions of metaplectic Whittaker functions also gives a hint as to where the difficulty in this project lies. The classical proof of Tokuyama’s formula is by induction on the rank using Pieri-rules: in the metaplectic setting, these have no convenient analogue. We sidestep this obstacle by refining the metaplectic statement to allow for a finer induction. Theorem 2 (the metaplectic version of Tokuyama’s theorem) is as a statement about the long element in the Weyl group, while the more general Theorem 1 is the same statement for any “beginning section” of a particular long word.

Phrasing a statement that lends itself to finer induction requires us to exploit interesting properties of the respective constructions. On the one hand, our joint work with Gautam Chinta and Paul E. Gunnells [CGP14] interprets the formulas of Chinta-Offen and McNamara [CO13, McN16] in terms of metaplectic Demazure-Lusztig operators. On the other

hand, one may exploit the branching structure of highest weight crystals of Dynkin type A to relate the formulas of Brubaker-Bump-Friedberg and McNamara [BBF11a, McN11] to smaller, similar expressions on Demazure crystals. The connection to Demazure-Lusztig operators is well motivated, and yields several avenues of possible applications; we mention these below.

1.2.1. Demazure operators. The relevance of classical Demazure and Demazure-Lusztig operators to the study of Whittaker functions was first indicated by work of Littelmann and Kashiwara [K⁺92] giving character formulas on a crystal, and of Brubaker, Bump and Licata [BBL14] relating them to Iwahori-fixed Whittaker functions. They have since been used by Patnaik [Pat14] to give a generalization of the Casselman-Shalika formula to Whittaker functions on the p -adic points of an affine Kac-Moody group.

The metaplectic versions of these operators were introduced by G. Chinta, P. E. Gunnells and the present author [CGP14]; the definitions involve the Chinta-Gunnells action of the Weyl group on rational functions over the weight lattice. This action was first used in [CG10] to construct p -parts of multiple Dirichlet series, and have since proved instrumental in metaplectic constructions, for example the ones mentioned above in 1.1.2.

1.3. Applications. We mention connections to the literature and avenues of further research utilizing the methods and results of this paper.

1.3.1. Iwahori-Whittaker functions. In [BBL14], the authors use Demazure and Demazure-Lusztig operators to compute values of Iwahori-Whittaker functions in terms of Hecke algebras, the geometry of Bott-Samelson varieties and the combinatorics of Macdonald polynomials. The analogies between these topics are intertwined with the combinatorics of the Bruhat order on the Weyl group, and identities satisfied by the Demazure and Demazure-Lusztig operators. Furthermore, the non-metaplectic version of the operator in Theorem 1 is related in [BBL14] to Iwahori-Whittaker functions. Recent work by Lee, Lenart, and Liu [LLL16] applies these results to compute coefficients of the transition matrix between natural bases of Iwahori-Whittaker functions. On the other hand, the work of Patnaik [Pat14] generalizing the Casselman-Shalika formula to the affine Kac-Moody setting also involves computing Iwahori-Whittaker functions, and their recursion in terms of Demazure-Lusztig operators. (The results of Brubaker-Bump-Licata and Patnaik are recalled in detail in section 9.)

The metaplectic Demazure and Demazure-Lusztig operators satisfy analogous identities to the classical, nonmetaplectic ones. Thus some of the above results will generalize to the metaplectic setting. Recent joint work with Manish Patnaik [PP15] shows that the connection between Iwahori-Whittaker functions and Demazure-Lusztig operators extends to the metaplectic setting (see section 9.3.3). It is natural to ask if the explicit crystal description of Iwahori-Whittaker functions given by Theorem 1 leads to a better understanding of all these results. It is especially interesting to consider how the connection with Iwahori-Whittaker functions, perhaps together with a more type-independent combinatorial description as mentioned in section 1.3.2 below, would elucidate the situation in the affine setting (see 1.3.3).s

1.3.2. The Alcove Path Model. The construction of Whittaker functions as a sum over the Weyl group [CO13, McN16] has the key feature that the Weyl group functional equations satisfied by the Whittaker function become very apparent. These functional equations play a key role in the analytic construction of global multiple Dirichlet series constructed

from the Whittaker functions. Moreover, they have proven useful in studying certain affine analogues.

The functional equations are less explicit in the description by crystal graphs. However, the crystal construction gives explicit formulas for individual coefficients of the Whittaker function. Reasons for trying to understand these coefficients are mentioned in section 1.3.3.

Various authors have worked on generalizing the crystal approach to root systems of other types: Chinta and Gunnells for type D [CG12], Beineke [BBF12], Brubaker, Bump, Chinta, Gunnells [BBCG12], Frechette, Friedberg and Zhang [FZ15] for type B and C , McNamara [McN11] working, less explicitly, with crystal bases in general. The resulting formulas are all significantly more intricate than the type A construction in [BBF11b].

A possible applications of this paper is to understand how the crystal approach extends to other types. Preliminary work by Beazley and Brubaker suggests that perhaps the alcove path model is better suited for creating a construction that generalizes the type A crystal approach. Demazure and Demazure-Lusztig operators promise to give a metaplectic Casselman-Shalika formula in terms of the alcove path model; we are currently investigating this avenue in joint work with Gautam Chinta, Cristian Lenart and Dan Orr. The resulting construction might better reflect the Weyl group symmetry of the individual coefficients of Whittaker functions.

1.3.3. Affine Weyl group multiple Dirichlet Series and metaplectic Whittaker functions. Recent work of Bucur-Diaconu [BD10], Lee-Zhang [LZ12] and Whitehead [Whi14] attempts to extend the theory of multiple Dirichlet series to the affine setting. There the theory of Eisenstein series is not (yet) available. These authors construct multiple Dirichlet series that satisfy functional equations corresponding to an affine Weyl group. The coefficients of these power series can be explicitly related to character sums and coefficients of L -functions [Whi14]. Some of our methods may lead to a combinatorial understanding of these coefficients. Furthermore, it would be especially interesting to understand the connection between affine Weyl group multiple Dirichlet series and Whittaker functions on p -adic points of affine Kac-Moody groups introduced by Patnaik [Pat14]. The possibility of extending the affine construction to the metaplectic setting is currently investigated by Patnaik and the author of this paper.

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2. PRELIMINARIES AND STATEMENT OF THE MAIN THEOREM

This section is dedicated to the statement of the main result of the paper (Theorem 1), and a brief outline of the methods and structure of the proof.

2.1. Background. We start by introducing some notation and background. We restrict ourselves to what is necessary to state the main theorem, Theorem 1, and to give an outline of the methods of the paper. Much of the background will be covered in more detail in later sections.

2.1.1. Notation. Let Λ be the weight lattice corresponding to a root system Φ of type A_r . We identify $\mathbb{C}(\Lambda)$ with a ring of rational functions $\mathbb{C}(\mathbf{x})$, where $\mathbf{x} = (x_1, \dots, x_{r+1})$ and $\mathbf{x}^{\alpha_1} = x_1/x_2$. The Weyl group W is generated by σ_i simple reflections. Let $w_0 \in W$ be the long element. We favour a particular reduced decomposition for w_0 (see (4.10) for this “favourite long word”); Theorem 1 is stated for elements $w \in W$ whose reduced decomposition is a “beginning segment” of this favourite long word (Definition 5). The integer n denotes the degree of the metaplectic cover of a split reductive algebraic group corresponding to Φ . We also introduce the indeterminate t , and $v = t^n$; in applications, we set $v = q^{-1}$, where q is the order of the residue field of a nonarchimedean local field.

2.1.2. Crystals and Gelfand-Tsetlin coefficients. The highest weight crystal $\mathcal{C}_{\lambda+\rho}$ and its parameterizations will be introduced in Section 4. For now, it suffices to say that it is a graph whose vertices are in bijection with a basis of the irreducible representation of highest weight $\lambda + \rho$, where $\lambda \in \Lambda$ is dominant and ρ is the Weyl vector. Vertices of a crystal can be parameterized by arrays of integers in various ways (using Gelfand-Tsetlin patterns, Γ -arrays, or Berenstein-Zelevinsky-Littelmann paths). To state Theorem 1 we need two functions on the vertices of a crystal $\mathcal{C}_{\lambda+\rho}$: the weight function $v \mapsto \text{wt}(v) \in \mathbb{Z}^{r+1}$ and the Gelfand-Tsetlin coefficient $v \mapsto G^{(n, \lambda+\rho)}(v) \in \mathbb{C}[t]$. This is the usual Gelfand-Tsetlin coefficient, described for nonmetaplectic and metaplectic cases in [BBF11b]. It depends on the degree n of the metaplectic cover via Gauss-sums. Furthermore, for every w beginning segment of the long word, we shall define $\mathcal{C}_{\lambda+\rho}^{(w)}$, the Demazure crystal corresponding to w . This is a subgraph of $\mathcal{C}_{\lambda+\rho}$ spanned by certain vertices depending on w (see Definition 6).

2.1.3. Demazure operators. Demazure operators \mathcal{D}_w and Demazure-Lusztig operators \mathcal{T}_w correspond to elements of the Weyl group, and act on $\mathbb{C}(\Lambda)$. The definitions of the non-metaplectic operators involve the natural action of the Weyl group W on $\mathbb{C}(\Lambda)$, inherited from the action of W on the weight lattice. In the metaplectic setting, this normal permutation action can be replaced by the Chinta-Gunnells action, and one may define metaplectic Demazure and Demazure-Lusztig operators, whose meaning depends on n . The definitions and properties are recalled - in the notation specific to type A_r - in section 3; these play a key role in the proof of Theorem 1.

2.1.4. Tokuyama’s theorem. Strictly speaking, this section is not necessary to understand the statement of Theorem 1; however, it provides motivation, and crucial guidance to the shape of the statement. As mentioned in section 1.1.3 generalizing Tokuyama’s theorem to the metaplectic setting and linking the constructions of metaplectic Whittaker functions is closely related; in fact the constructions give rise to the statement of a metaplectic version.

We explain this briefly here; the theorem will be discussed in detail in Section 5. Tokuyama's theorem is a deformation of the Weyl character formula in type A :

$$(2.1) \quad \mathbf{x}^\rho \cdot \prod_{\alpha \in \Phi^+} (1 - v \cdot \mathbf{x}^\alpha) \cdot s_\lambda(\mathbf{x}) = \sum_{b \in \mathcal{C}_{\lambda+\rho}} G(b) \cdot \mathbf{x}^{\text{wt}(b)},$$

where s_λ is the Schur function. The left hand side essentially agrees with the Casselman-Shalika formula for Whittaker functions (with the deforming parameter v specialized to q^{-1}). The Schur function can be expressed by the Weyl character formula as

$$(2.2) \quad s_\lambda(\mathbf{x}) = \frac{1}{\mathbf{x}^\rho \cdot \prod_{\alpha \in \Phi^+} (1 - \mathbf{x}^\alpha)} \cdot \sum_{w \in S_r} (-1)^{\ell(w)} \cdot \mathbf{x}^{w(\lambda+\rho)}.$$

Chinta-Offen [CO13] show what a correct metaplectic analogue of the right hand side in (2.2) is, replacing the action of the Weyl group W on $\mathbb{C}(\Lambda)$ by the Chinta-Gunnells metaplectic action. One may use the results of [CGP14] to reformulate the “left-hand side” in terms of Demazure-Lusztig operators, i.e. as

$$(2.3) \quad \sum_{u \in W} \mathcal{T}_u$$

acting on a monomial. The necessary background will be covered in detail in Section 3.

On the right hand side of (2.1), the Gelfand-Tsetlin coefficients $G(b) = G^{(1, \lambda+\rho)}(b)$ appear. This reproduces the construction of the same Whittaker function as a sum over a crystal base (Brubaker-Bump-Friedberg [BBF11a]) in both the nonmetaplectic case (for $n = 1$) and the metaplectic setting (for higher n).

2.2. Statement of Main Theorem. The main result of the paper is a crystal description of sums of Demazure-Lusztig operators in type A . More precisely, we prove the following.

Theorem 1. *Let $\lambda = (\lambda_1, \dots, \lambda_r, \lambda_{r+1})$ be any dominant, effective weight, $\rho = (r, r-1, \dots, 1, 0)$, w a beginning section of the long word. Then*

$$(2.4) \quad \left(\sum_{u \leq w} \mathcal{T}_u \right) \mathbf{x}^{w_0(\lambda)} = \mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n, \lambda+\rho)}(v) \cdot \mathbf{x}^{\text{wt}(v)}.$$

Here \leq is the Bruhat order, $G^{(n, \lambda+\rho)}(v)$ is the Gelfand-Tsetlin coefficient corresponding to v (by Definition 4), and $\mathcal{C}_{\lambda+\rho}^{(w)}$ is the Demazure-crystal corresponding to w (as is Definition 6).

The statement (2.4) provides the combinatorial link between the approaches to constructing metaplectic Whittaker functions described in section 1.1.2. The special case of this statement for $w = w_0$ and $n = 1$ is exactly Tokuyama's theorem (See section 5). The statement is formally stronger than Tokuyama's theorem even in the nonmetaplectic setting, and provides a metaplectic analogue for higher n . We state this analogue on its own.

Theorem 2. *(Tokuyama's Theorem, Metaplectic Version.) Let $\lambda = (\lambda_1, \dots, \lambda_{r+1})$ be any dominant, effective weight and $\rho = (r, \dots, 1, 0)$. Then*

$$(2.5) \quad \left(\sum_{u \in W} \mathcal{T}_u \right) (\mathbf{x}^{w_0(\lambda)}) = \mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}} G^{(n, \lambda+\rho)}(v) \cdot \mathbf{x}^{\text{wt}(v)}.$$

This special case of the identity is present when work of Brubaker-Bump-Friedberg-Hoffstein [BBF11a], Chinta-Gunnells-Offen [CG10, CO13], and McNamara [McN11, McN16] are combined, but Theorem 1 provides a much more direct connection. In addition, as mentioned in 1.3.1, the operators

$$(2.6) \quad \sum_{u \leq w} \mathcal{T}_u$$

are related to the construction of Iwahori-Whittaker functions; in this sense Theorem 1 may be interpreted as a crystal description of Iwahori-Whittaker functions. We shall make the connection between Theorem 1 and Whittaker and Iwahori-Whittaker functions more explicit in section 9.

2.3. Methods and Outline. We give an overview of the methods and structure of the proof of Theorem 1.

Tokuyama's proof of the identity (2.1) uses Pieri rules, i.e. is by induction on the rank r of the type A_r of the root system Φ . Pieri rules are not available in the metaplectic setting, so instead we “refine” the induction. Theorem 1 interprets Tokuyama's formula in type A_r as a statement about the (favourite) long word $w_0^{(r)}$. This is the following reduced decomposition of the long element.

$$w_0 = w_0^{(r)} = \sigma_1 \sigma_2 \sigma_1 \cdots \sigma_{r-1} \cdots \sigma_1 \sigma_r \cdots \sigma_1.$$

This word has the property that it begins with $w_0^{(r-1)}$, the favourite long word in type A_{r-1} . Since Theorem 1 formulates an identity for every beginning section of the word $w_0^{(r)}$, if we want to prove this statement by induction, the step from A_{r-1} to A_r is now not one step, but r “smaller” steps.

This is the main idea - the proof is in fact by induction; the main tools are branching properties of type A highest weight crystals, and identities of Demazure(-Lusztig) operators.

2.3.1. Structure of the induction. The edges of a highest weight crystal $\mathcal{C}_{\lambda+\rho}$ of type A_r are labelled by indices $1, 2, \dots, r$. Removing the edges labelled with r breaks the crystal into the disjoint union of highest weight crystals of type A_{r-1} . The same is true for a Demazure crystal $\mathcal{C}_{\lambda+\rho}^{(w)}$ as long as w is a beginning section of the favourite long word $w_0^{(r)}$. This fact is crucial to the proof. Let w be a beginning section of long word $w_0^{(r)}$ that is *not* a beginning section of $w_0^{(r-1)}$, i.e. has the form

$$w = w_0^{(r-1)} \sigma_r \cdots \sigma_{r-k},$$

where $k = \ell(w) - \ell(w_0^{(r-1)}) - 1$. Call the statement of Theorem 1 for this particular w and fixed n (but for any λ) $IW_{r,k}^{(n)}$. For $k = r - 1$, i.e. $w = w_0^{(r)}$, Theorem 1 specializes to Theorem 2, and thus we use the notation $Tok_r^{(n)} = IW_{r,r-1}^{(n)}$.

It follows from the branching property of Demazure crystals mentioned above, as well as from a simple fact about the Bruhat order (Lemma 18) that $IW_{r,k}^{(n)}$ can be reduced to $Tok_{r-1}^{(n)}$, and statements describing the action of simpler operators on a monomial. The full reduction argument will be explained later; for now we only say that in addition to $IW_{r,k}^{(n)}$ and $Tok_r^{(n)}$, some auxiliary statements will be phrased: $M_{r,k}^{(n)}$ and $N_{r,k}^{(n)}$, and (the special case) $N_{r,r-1}^{(n)}$. The statement $N_{r,k}^{(n)}$, for example, concerns the action of the operator

$T_r T_{r-1} \dots T_{r-k}$ on a monomial. We show in Section 7 that to prove $IW_{r,k}^{(n)}$ for any r and any $0 \leq k < r$ it suffices to prove the statement $N_{r,r-1}^{(n)}$ for any r . This reduction of $IW_{r,k}^{(n)}$ to $N_{r,r-1}^{(n)}$ essentially follows from the branching of Demazure crystals and Gelfand-Tsetlin coefficients, and some properties of the metaplectic Demazure-Lusztig operators.

By the end of Section 7, the only thing remaining from the proof of Theorem 1 is to prove $N_{r,r-1}^{(n)}$, a statement about the action of the operator $T_r T_{r-1} \dots T_1$ on a monomial. This statement is proved by a (somewhat technical) induction in Section 8, with a rank one auxiliary computation included in Appendix A.

2.3.2. Outline. The necessary background is summarized in three sections. Section 3 explains the Chinta-Gunnells action and metaplectic Demazure operators; Section 4 describes parameterizations and branching of type A highest weight crystals and contains the definition of Gelfand-Tsetlin coefficients; Section 5 contains the re-phrasing of Tokuyama's result into the language of Demazure-Lusztig operators and crystals.

The proof of Theorem 1 spans three sections. Section 6 is preparation: it defines Demazure crystals and examines the Gelfand-Tsetlin coefficients on these in terms of the branching properties discussed in Section 4. Some helpful conventions, designed to make the notation of the proof lighter, are also introduced here. Section 7 contains the proof of Theorem 1 through reduction to a sequence of simpler statements (from $IW_{r,k}^{(n)}$ to $N_{r,r-1}^{(n)}$), as explained above. The final statement of the sequence, $N_{r,r-1}^{(n)}$ is then proved in Section 8 (and Appendix A).

Section 9 relates Theorem 1 to metaplectic Whittaker functions and Iwahori-Whittaker functions. The constructions mentioned in the Introduction are recalled in a little more detail to demonstrate how the formulas line up with the expressions in Theorem 1.

3. METAPLECTIC DEMAZURE AND DEMAZURE-LUSZTIG OPERATORS

Theorem 1 describes the action of metaplectic Demazure-Lusztig operators on a monomial. As mentioned in 1.2.1 the metaplectic analogues of the classical Demazure and Demazure-Lusztig operators were introduced in [CGP14]. In this section, we briefly review the results of that paper, specializing to type A root systems. The definition, elementary properties, and the identities Theorem 6 and Theorem 7 will be necessary for the proof. The metaplectic operators are built on the Chinta-Gunnells action; we recall the definition in Section 3.2. We restrict our attention to type A , hence some of the machinery that is necessary in [CGP14] can be spared.

3.1. Notation. The following is standard notation for root systems and the Weyl group. The reader may refer to [Hum78] as a source.

Let Φ be an irreducible reduced root system of type A_r with Weyl group W . We may view Φ as embedded into \mathbb{R}^{r+1} . Let e_1, \dots, e_{r+1} denote the standard basis of \mathbb{R}^{r+1} , and take

$$\Phi = \{e_i - e_j \in \mathbb{R}^{r+1} \mid 1 \leq i \neq j \leq r+1\}.$$

Let $\Phi = \Phi^+ \cup \Phi^-$ be the decomposition into positive and negative roots ($e_i - e_j \in \Phi^+$ if $i < j$). Let $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be the set of simple roots; $\alpha_i = e_i - e_{i+1}$ ($1 \leq i \leq r$). and let σ_i be the Weyl group element corresponding to the reflection through the hyperplane perpendicular to α_i . Set

$$(3.1) \quad \Phi(w) = \{\alpha \in \Phi^+ : w(\alpha) \in \Phi^-\}.$$

Consider the weight lattice

$$\Lambda = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \lambda_{r+1}) \in \mathbb{Z}^{r+1}\};$$

then $\Lambda \subset \mathbb{R}^{r+1}$ contains Φ as a subset. Let $\mathcal{A} = \mathbb{C}[\Lambda]$ be the ring of Laurent polynomials on Λ . Let \mathcal{K} be the field of fractions of \mathcal{A} . The action of W on the lattice Λ induces an action of W on \mathcal{K} : we put

$$(3.2) \quad (w, x^\lambda) \mapsto x^{w\lambda} =: w.x^\lambda,$$

and then extend linearly and multiplicatively to all of \mathcal{K} . We denote this action using the lower dot $(w, f) \mapsto w.f$ (to distinguish it from the metaplectic W -action on \mathcal{K} constructed below in (3.10)) and refer to this as the “nonmetaplectic” group action.

Let $\mathbf{x} = (x_1, \dots, x_r, x_{r+1})$. We may identify \mathcal{K} with $\mathbb{C}(x_1, \dots, x_{r+1}) = \mathbb{C}(\mathbf{x})$ by writing $x_i = \mathbf{x}^{e_i}$. In general, for $\lambda = \sum_i \lambda_i e_i \in \Lambda$ as above, we write $\mathbf{x}^\lambda = x_1^{\lambda_1} \cdot x_2^{\lambda_2} \cdots x_{r+1}^{\lambda_{r+1}}$. Note that the Weyl group $W \cong S_{r+1}$, and the nonmetaplectic action (3.2) of σ_i on $\mathbb{C}(\mathbf{x})$ is by swapping x_i and x_{i+1} .

The definition of the Chinta-Gunnells action in [CGP14] requires a W -invariant \mathbb{Z} -valued quadratic form Q defined on Λ , which defines a bilinear form $B(\alpha, \beta) = Q(\alpha + \beta) - Q(\alpha) - Q(\beta)$. Fix a positive integer n ; n determines a collection of integers $\{m(\alpha) : \alpha \in \Phi\}$ by

$$(3.3) \quad m(\alpha) = n / \gcd(n, Q(\alpha)),$$

and a sublattice $\Lambda_0 \subset \Lambda$ by

$$(3.4) \quad \Lambda_0 = \{\lambda \in \Lambda : B(\alpha, \lambda) \equiv 0 \pmod{n} \text{ for all simple roots } \alpha\}.$$

In type A , we may give an explicit example of a $W \cong S_{r+1}$ -invariant, integer-valued quadratic form. For $\lambda \in \Lambda$, (and c arbitrary), let

$$(3.5) \quad Q(\lambda) = - \sum_{h < j} \lambda_h \lambda_j - c \cdot \left(\sum_{h=1}^{r+1} \lambda_h \right)^2 = -(1+2c) \cdot \sum_{h < j} \lambda_h \lambda_j - c \cdot \sum_{h=1}^{r+1} \lambda_h^2.$$

Then certainly Q (and thus B) are integer valued on Λ and $n\Lambda \subseteq \Lambda_0$. Furthermore, it is easy to check that $Q(\alpha_i) = 1$ and $B(\alpha_i, \lambda) = \lambda_i - \lambda_{i+1}$. This implies that the sublattice Λ_0 is

$$(3.6) \quad \Lambda_0 = \{\lambda \in \mathbb{Z}^{r+1} : \lambda_i \equiv \lambda_j \pmod{n} \text{ for all } 1 \leq i, j \leq r+1\}.$$

Since all roots are of the same length, $m(\alpha) = n / \gcd(n, Q(\alpha))$ is the same for every root. In particular, with the choice of Q above, $Q(\alpha) = 1$ and hence $m(\alpha) = n$.

Remark 1. For any simple root α , we have $m(\alpha)\alpha = n\alpha \in \Lambda_0$. (This is a special case of [CGP14, Lemma 1].)

3.2. The Chinta-Gunnells action. The Chinta-Gunnells action is a “metaplectic” action of a Weyl group on a ring of rational functions; the action depends on the metaplectic degree. We use the same definition as in [CGP14], which in turn is the same as the one defined in Chinta-Gunnells [CG10] and specializes to the type A action in Chinta-Offen [CO13].

Following [CGP14, Section 2], let $\lambda \mapsto \bar{\lambda}$ be the projection $\Lambda \rightarrow \Lambda/\Lambda_0$ and $(\Lambda/\Lambda_0)^*$ be the group of characters of the quotient lattice. Any $\xi \in (\Lambda/\Lambda_0)^*$ induces a field automorphism of \mathcal{K}/\mathbb{C} by setting $\xi(x^\lambda) = \xi(\bar{\lambda}) \cdot x^\lambda$ for $\lambda \in \Lambda$. This leads to the direct sum decomposition

$$(3.7) \quad \mathcal{K} = \bigoplus_{\bar{\lambda} \in \Lambda/\Lambda_0} \mathcal{K}_{\bar{\lambda}}$$

where $\mathcal{K}_{\bar{\lambda}} = \{f \in \mathcal{K} : \xi(f) = \xi(\bar{\lambda}) \cdot f \text{ for all } \xi \in (\Lambda/\Lambda_0)^*\}$.

The next ingredient is a set of complex parameters v, g_0, \dots, g_{n-1} satisfying

$$(3.8) \quad g_0 = -1 \text{ and } g_i g_{n-i} = v^{-1} \text{ for } i = 1, \dots, n-1.$$

For all other j we define $g_j := g_{r_n(j)}$, where $0 \leq r_n(j) < n-1$ denotes the remainder upon dividing j by n . The parameters g_0, \dots, g_{n-1} will be chosen in section 3.3 to be Gauss sums.

We may now recall the definition of the metaplectic action of the Weyl group W on \mathcal{K} .

Definition 1. [CGP14, Section 2, (7)] For $f \in \mathcal{K}_{\bar{\lambda}}$ and the generator $\sigma_\alpha \in W$ corresponding to a simple root α , define

$$(3.9) \quad \sigma_i(f) = \frac{\sigma_i \cdot f}{1 - vx^{m(\alpha_i)\alpha_i}} \cdot \left[x^{-r_m(\alpha_i) \left(-\frac{B(\lambda, \alpha_i)}{Q(\alpha_i)} \right) \cdot \alpha_i} \cdot (1 - v) \right. \\ \left. - v \cdot g_{Q(\alpha_i) - B(\lambda, \alpha_i)} \cdot x^{(1-m(\alpha_i))\alpha_i} \cdot (1 - x^{m(\alpha_i)\alpha_i}) \right]$$

where λ is any lift of $\bar{\lambda}$ to Λ . Here the quantity in brackets depends only on $\bar{\lambda}$. We extend the definition of σ_α to \mathcal{K} by additivity. Then (1) extends to an action of the full Weyl group W on \mathcal{K} , which we denote

$$(3.10) \quad (w, f) \mapsto w(f).$$

Using the notation specific to type A , Definition 1 can be rewritten for $f = \mathbf{x}^\lambda$ as follows.

$$(3.11) \quad \sigma_i(f) = \frac{\sigma_i \cdot f}{1 - v \cdot \left(\frac{x_i}{x_{i+1}} \right)^n} \cdot \left[\left(\frac{x_i}{x_{i+1}} \right)^{-r_n(\lambda_{i+1} - \lambda_i)} \cdot (1 - v) \right. \\ \left. - v \cdot g_{1+\lambda_{i+1}-\lambda_i} \cdot \left(\frac{x_i}{x_{i+1}} \right)^{1-n} \cdot \left(1 - \left(\frac{x_i}{x_{i+1}} \right)^n \right) \right].$$

The following Lemma is crucial in computations; it is used repeatedly, if implicitly, in the proof of Theorem 1. It relies on the fact that the quantity in brackets in (1) depends only on $\bar{\lambda}$ and not λ .

Lemma 3. [CGP14, Lemma 2] Let $f \in \mathcal{K}$ and $h \in \mathcal{K}_0$. Then for any $w \in W$,

$$w(hf) = (w.h) \cdot w(f).$$

Here $w.h$ means the non-metaplectic action, while \cdot denotes multiplication in \mathcal{K} . □

The significance of Lemma 3 is due to the fact that the action of W on \mathcal{K} defined by (1), though \mathbb{C} -linear, is *not* by endomorphisms of that ring, i.e. it is not in general multiplicative. The point of Lemma 3 is that if we have a product of two terms hf , the first of which satisfies $h \in \mathcal{K}_0$ (e.g. the exponents of h are divisible by n), then we can apply w to the product hf by performing the usual permutation action on h and then acting on f by the metaplectic W -action.

The following lemma shows a symmetric monomial with respect to σ_i . It will be of use in computations. We omit the (straightforward) proof.

Lemma 4. $\sigma_i(x_i^{a+1}x_{i+1}^{a+n}) = x_i^{a+1}x_{i+1}^{a+n}$ for every n .

3.3. Gauss sums. The complex parameters v, g_0, \dots, g_{n-1} of the Chinta-Gunnells action are chosen to be Gauss sums in applications. Similar Gauss sums (g^b and h^b) are used in [BBF11b] to define Gelfand-Tsetlin coefficients on a crystal graph (see section 4).

We make the choice of parameters explicit here. We start by describing the functions g^b and h^b , following [BBF11b, Chapter 1] for notation and definitions. We will then choose the parameters v, g_0, \dots, g_{n-1} to satisfy the conditions of (3.8). For facts about the power residue symbol we refer the reader to [BBF06].

3.3.1. Notation. Let F be an algebraic number field containing the group μ_{2n} of $2n$ -th roots of unity. Let S be a finite set of places of F , large enough that it contains all the places that are Archimedean or ramified over \mathbb{Q} , and the ring of S -integers $\mathfrak{o}_S = \{x \in F \mid |x|_v \leq 1 \text{ for } v \notin S\}$ is a principal ideal domain. Let ψ be a character on F_S of conductor \mathfrak{o}_S . For any $m, c \in \mathfrak{o}_S$, $c \neq 0$, consider the n -th power residue symbol $\left(\frac{m}{c}\right)_n$. Recall that $\left(\frac{m}{c}\right)_n$ is zero unless m is prime to c . It is multiplicative, i. e. $\left(\frac{m}{c}\right)_n \cdot \left(\frac{m}{b}\right)_n = \left(\frac{m}{bc}\right)_n$. If p is a prime and m is coprime to p , then $\left(\frac{m}{p}\right)_n$ is the element of μ_n satisfying $\left(\frac{m}{p}\right)_n \equiv m^{\frac{n-1}{n}} \pmod{p}$.

With the notation above, define the Gauss sum

$$(3.12) \quad g(m, c) = \sum_{a \bmod c} \left(\frac{a}{c}\right)_n \psi\left(\frac{am}{c}\right).$$

Fix a p prime in \mathfrak{o}_S , and let q be the cardinality of the residue field $\mathfrak{o}_S/p\mathfrak{o}_S$. We assume $q \equiv 1 \pmod{2n}$. Define $g(a) = g(p^{a-1}, p^a)$ and $h(a) = g(p^a, p^a)$ for any $a > 0$. In this case we have

$$g(a) = \sum_{b \bmod p^a} \left(\frac{b}{p^a}\right)_n \psi\left(\frac{b}{p}\right) = q^{a-1} \cdot \sum_{b \bmod p} \left(\frac{b}{p}\right)_n^a \psi\left(\frac{b}{p}\right)$$

and

$$h(a) = \sum_{b \bmod p^a} \left(\frac{b}{p^a}\right)_n \psi(b) = q^{a-1} \cdot \sum_{b \bmod p} \left(\frac{b}{p}\right)_n^a \cdot 1 = \begin{cases} 0 & n \nmid a; \\ (q-1) \cdot q^{a-1} & n \mid a. \end{cases}$$

3.3.2. Choice of parameters. We are ready to define the functions g^b and h^b . These appear in Section 4 in the definition of Gelfand-Tsetlin coefficients, and the proof of Theorem 1 depends on computations that use g^b and h^b . Let

$$(3.13) \quad g^b(a) = q^{-a} \cdot g(a) \quad \text{and} \quad h^b(a) = q^{-a} \cdot h(a).$$

The following identities imply that the value of both $g^b(a)$ and $h^b(a)$ only depend on the residue of a modulo n .

$$(3.14) \quad h^b(a) = \begin{cases} 0 & n \nmid a; \\ 1 - \frac{1}{q} & n \mid a. \end{cases} \quad \text{and} \quad g^b(a) = q^{-1} \cdot \sum_{b \bmod p} \left(\frac{b}{p}\right)_n^a \psi\left(\frac{b}{p}\right).$$

If a is divisible by n then

$$(3.15) \quad g^b(a) = -q^{-1},$$

and if $0 < a < n$ then

$$(3.16) \quad g^b(a) \cdot g^b(n-a) = q^{-1}.$$

Recall the conditions (3.8) imposed on the parameters v, g_0, \dots, g_{n-1} . The parameters must satisfy $g_0 = -1$ and $g_i g_{n-i} = v^{-1}$ for $1 \leq i \leq n-1$. We can choose these parameters by modifying the functions g^b and h^b . Take $v = q^{-1}$ and

$$(3.17) \quad g_i = v^{-1} \cdot g^b(i) = q \cdot g^b(i) = \sum_{b \bmod p} \left(\frac{b}{p}\right)_n^i \psi\left(\frac{b}{p}\right) \text{ for } i = 1, \dots, n-1.$$

Then (3.15) implies $g_0 = q \cdot (-q^{-1}) = -1$ and (3.16) implies

$$g_i g_{n-i} = v^{-2} \cdot g^b(i) \cdot g^b(n-i) = v^{-2} \cdot v = v^{-1}.$$

We summarize the choices of parameters in the following claim. The notation $t^n = v = q^{-1}$ is introduced for later convenience.

Claim 5. *If $n \nmid a$, then $h^b(a) = 0$, and*

$$v \cdot g_a = q^{-1} \cdot \mathfrak{g}^\psi(-a) = \gamma(a) = g^b(a) = q^{-1} \cdot \sum_{b \bmod p} \left(\frac{b}{p}\right)_n^a \psi\left(\frac{b}{p}\right).$$

However, if $n|a$, then $h^b(a) = 1 - v$, $\gamma(a) = g_a = g_0 = -1$, and $g^b(a) = -q^{-1} = -v = -t^n$.

3.4. Metaplectic Demazure and Demazure-Lusztig operators. The definitions below follow [CGP14], making use of the identification of \mathcal{K} and $\mathbb{C}(\mathbf{x})$ and the Chinta-Gunnells action introduced in section 3.2. Both the Demazure operators and the Demazure-Lusztig operators are divided difference operators on \mathcal{K} .

Let $1 \leq i \leq r$ and $f \in \mathbb{C}(\mathbf{x})$. We define the *Demazure operators* by

$$(3.18) \quad \mathcal{D}_i(f) = \mathcal{D}_{\sigma_i}(f) = \frac{f - \mathbf{x}^{n\alpha_i} \cdot \sigma_i(f)}{1 - \mathbf{x}^{n\alpha_i}},$$

and the *Demazure-Lusztig operators* by

$$(3.19) \quad \begin{aligned} \mathcal{T}_i(f) &= \mathcal{T}_{\sigma_i}(f) = (1 - v \cdot \mathbf{x}^{n\alpha_i}) \cdot \mathcal{D}_i(f) - f \\ &= (1 - v \cdot \mathbf{x}^{n\alpha_i}) \cdot \frac{f - \mathbf{x}^{n\alpha_i} \cdot \sigma_i(f)}{1 - \mathbf{x}^{n\alpha_i}} - f. \end{aligned}$$

Recall that here $\mathbf{x}^{n\alpha_i}$ is shorthand for $\frac{x_i^n}{x_{i+1}^n}$. When there is no danger of confusion, we write more simply

$$\mathcal{D}_i = \frac{1 - \mathbf{x}^{n\alpha_i} \sigma_i}{1 - \mathbf{x}^{n\alpha_i}} \quad \text{and} \quad \mathcal{T}_i = (1 - v \cdot \mathbf{x}^{n\alpha_i}) \cdot \mathcal{D}_i - 1,$$

that is, a rational function h in the above equations is interpreted to mean the “multiplication by h ” operator. The rational functions here are in \mathcal{K}_0 (see Remark 1).

The operators \mathcal{D}_i and \mathcal{T}_i satisfy the same braid relations as the σ_i [CGP14, Proposition 7.]. Consequently, one may define \mathcal{D}_w and \mathcal{T}_w for any $w \in W$: let $w = \sigma_{i_1} \cdots \sigma_{i_l}$ be a reduced expression for w in terms of simple reflections. Then

$$\mathcal{D}_w := \mathcal{D}_{i_1} \cdots \mathcal{D}_{i_l} \quad \text{and} \quad \mathcal{T}_w := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_l}.$$

We also introduce a metaplectic analogue of the Weyl denominator. Let

$$(3.20) \quad \Delta_v = \Delta_v^{(n)} = \prod_{\alpha \in \Phi^+} (1 - v \cdot \mathbf{x}^{n\alpha}).$$

If $v = 1$ we write simply $\Delta_v = \Delta$. Now we are ready to state the metaplectic Demazure formula and Demazure-Lusztig formula. (As before, the notation is specific to type A)

Theorem 6. [CGP14, Theorem 3.] For the long element w_0 of the Weyl group W we have

$$\mathcal{D}_{w_0} = \frac{1}{\Delta} \cdot \sum_{w \in W} \text{sgn}(w) \cdot \prod_{\alpha \in \Phi(w^{-1})} \mathbf{x}^{n\alpha} \cdot w.$$

Theorem 7. [CGP14, Theorem 4.] We have

$$\Delta_v \cdot \mathcal{D}_{w_0} = \sum_{w \in W} \mathcal{T}_w.$$

The following technical lemmas about polynomials annihilated by Demazure operators are of use in the proof of Theorem 1.

Lemma 8. We have the following.

- (i) A polynomial f is annihilated by $\mathcal{D}_i = \mathcal{D}_{\sigma_i}$ if and only if $\sigma_i(x_{i+1}^n \cdot f) = x_{i+1}^n \cdot f$.
- (ii) If $\mathcal{D}_w(g) = 0$ for some w in the Weyl group W , and w_0 is the long element of W , then $\mathcal{D}_{w_0}g = 0$.

Proof. The proof of (i) is obvious from the definition of \mathcal{D}_i and Lemma 3. For (ii), let $u = w_0w^{-1}$, so that $w_0 = u \cdot w$. Since w_0 is the longest element, we have $\ell(w_0) = \ell(u) + \ell(w)$, and as a consequence $\mathcal{D}_{w_0} = \mathcal{D}_u \circ \mathcal{D}_w$. Thus $\mathcal{D}_{w_0}g = \mathcal{D}_u(\mathcal{D}_wg) = \mathcal{D}_u(0) = 0$. \square

The following is a trivial corollary of Lemmas 8 and 4, as the action of σ_i only involves the exponents of x_i and x_{i+1} .

Corollary 9. If $\beta = (\beta_1, \dots, \beta_{r+1})$ and $\beta_i = \beta_{i+1} + 1$, then $\mathcal{D}_i(\mathbf{x}^\beta) = 0$.

4. HIGHEST WEIGHT CRYSTALS AND GELFAND-TSETLIN PATTERNS

We turn our attention to the “crystal side” of Theorem 1: a sum whose terms involve Gelfand-Tsetlin coefficients, and are summed over a crystal. In this section, we present a primer on objects in this picture. Crystals can be parametrized in more than one way, we shall see that moving back and forth between parameterizations is not particularly difficult, hence one may choose the language that is most convenient in any given context. Gelfand-Tsetlin coefficients are defined in terms of these parameterizations.

We describe, in turn, highest weight crystals (4.1), Gelfand-Tsetlin patterns with “ Γ -arrays” (4.2), and Berenstein-Zelevinsky-Littelmann paths (4.3). Gelfand-Tsetlin patterns are arrays of integers; the ones with a fixed top row are in bijection with vertices of a highest weight crystal. The bijection is via Berenstein-Zelevinsky-Littelmann paths, and the Γ -array corresponding to a pattern. The precise statement of this bijection is the content of Proposition 10. Gelfand-Tsetlin coefficients are defined in section 4.4. Finally, in section 4.5, we recall the branching property of type A highest-weight crystals. This will be revisited for Demazure crystals in Section 6, and is a crucial ingredient in the proof of Theorem 1.

Throughout the section, we follow the presentation of Chapter 2 of Brubaker-Bump-Friedberg [BBF11b], in less detail. We (implicitly) rely on other sources as well. In particular, for the combinatorial definition of a crystal graph, we use Hong-Kang [HK02] and Kashiwara [Kas95]. For the correspondence between Gelfand-Tsetlin patterns and highest weight crystals, Berenstein-Zelevinsky [BZ93, BZ⁺96], Littelmann [Lit98], or Lusztig [Lus90] are further references.

4.1. Highest weight crystals. The general definition of a crystal can be found in Kashiwara [Kas95]. Here we only consider type A highest weight crystals.

Recall the notation introduced in section 3.1 for root systems of type A_r . In particular, recall that the weight lattice Λ is identified with \mathbb{Z}^{r+1} ; α_i are the simple roots for $1 \leq i \leq r$. Let $h_i = \mathbf{e}_i^* - \mathbf{e}_{i+1}^* \in \Lambda^*$ where $\mathbf{e}_1^*, \mathbf{e}_2^*, \dots, \mathbf{e}_{r+1}^*$ denotes the standard dual basis of \mathbb{R}^{r+1} . (We use \mathbf{e}_i^* here to distinguish the basis vectors \mathbf{e}_i from the Kashiwara operators e_i below.) We have $(\cdot, \cdot) : \Lambda \times \Lambda \rightarrow \mathbb{Q}$ a bilinear symmetric form, and let $\langle \cdot, \cdot \rangle : \Lambda^* \times \Lambda \rightarrow \mathbb{Z}$ denote the canonical pairing. Note that $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$, $\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for $i \in I$ and $\lambda \in P$, and $(\alpha_i, \alpha_j) \leq 0$ for $i, j \in I$, $i \neq j$.

A type A_r crystal \mathcal{C} is a set \mathbf{B} endowed with a weight function $\text{wt} : \mathbf{B} \rightarrow \Lambda$, functions $\varepsilon_i : \mathbf{B} \rightarrow \mathbb{Z} \sqcup \{-\infty\}$, $\varphi_i : \mathbf{B} \rightarrow \mathbb{Z} \sqcup \{-\infty\}$ and Kashiwara operators $e_i : \mathbf{B} \rightarrow \mathbf{B} \sqcup \{0\}$, $f_i : \mathbf{B} \rightarrow \mathbf{B} \sqcup \{0\}$ for every $1 \leq i \leq r$. Elements of \mathbf{B} are called elements or vertices of the crystal. A crystal satisfies the following axioms. (Let $-\infty + n = -\infty$ for every $n \in \mathbb{Z}$.)

(i) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle$ for every $1 \leq i \leq r$

(ii) If $e_i(b) \neq 0$, then

$$\begin{aligned} \varepsilon_i(e_i(b)) &= \varepsilon_i(b) - 1, \\ \varphi_i(e_i(b)) &= \varphi_i(b) + 1, \\ \text{wt}(e_i(b)) &= \text{wt}(b) + \alpha_i. \end{aligned}$$

(iii) If $f_i(b) \neq 0$, then

$$\begin{aligned} \varepsilon_i(f_i(b)) &= \varepsilon_i(b) + 1, \\ \varphi_i(f_i(b)) &= \varphi_i(b) - 1, \\ \text{wt}(f_i(b)) &= \text{wt}(b) - \alpha_i. \end{aligned}$$

(iv) For $b_1, b_2 \in \mathbf{B}$, we have $b_2 = f_i(b_1)$ if and only if $b_1 = e_i(b_2)$.

(v) If $\varphi_i(b) = -\infty$, then $e_i(b) = f_i(b) = 0$.

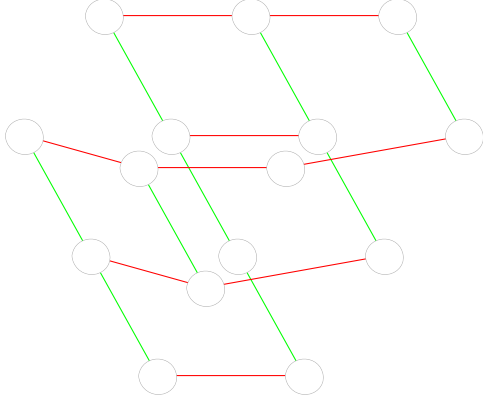
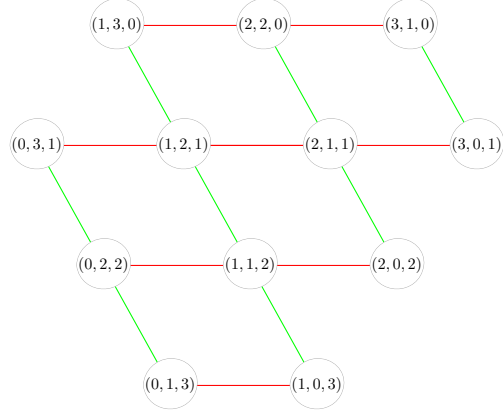
Recall that the weight $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \lambda_{r+1})$ is called dominant if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r+1}$; strongly dominant if $\lambda_1 > \lambda_2 > \dots > \lambda_{r+1}$; λ is effective if $\lambda_{r+1} \geq 0$. There is a partial ordering on \mathbb{Z}^{r+1} where $\mu \preceq \lambda$ if and only if $\lambda - \mu$ lies in the cone generated by simple roots. For every dominant weight λ there is a corresponding crystal graph \mathcal{C}_λ with highest weight λ . The function wt maps the vertices of \mathcal{C}_λ to weights of the representation V_λ of $\mathfrak{gl}_{r+1}(\mathbb{C})$ of highest weight λ . The Kashiwara operators determine a directed graph structure on \mathcal{C}_λ : there is an edge $v \xrightarrow{i} w$ if and only if $f_i(v) = w \neq 0$. We say this edge is labeled with i . The number of vertices in \mathcal{C}_λ with weight μ is equal to the multiplicity of the weight μ in the representation V_λ . In particular, \mathcal{C}_λ has exactly one element v_{highest} with weight λ . If w_0 denotes the longest element of the type Weyl group $W \cong S_{r+1}$, then $w_0\lambda = (\lambda_{r+1}, \lambda_r, \dots, \lambda_2, \lambda_1)$, and \mathcal{C}_λ has exactly one element v_{lowest} with weight $w_0\lambda$ (this is the “lowest” element).

The edges labelled with the same index i (for $1 \leq i \leq r$) determine disjoint “ i -strings” in the crystal. These are themselves isomorphic to type A_1 highest weight crystals. The functions ε_i and φ_i determine where a vertex is within an i -string:

$$\varepsilon_i(b) = \max\{n \geq 0 \mid e_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \geq 0 \mid f_i^n b \neq 0\}.$$

We conclude this section by an example.

Example 1. Figure 1 shows a crystal of type A_2 corresponding to highest weight $(3, 1, 0)$. The red edges correspond to the label 1, the green edges to the label 2. Figure 2 shows the image of the same crystal under the weight map.

FIGURE 1. The crystal $\mathcal{C}_{(3,1,0)}$.FIGURE 2. The weights of the $\mathfrak{gl}_3(\mathbb{C})$ representation of highest weight $(3,1,0)$.

4.2. Gelfand-Tsetlin patterns. We recall the definition of Gelfand-Tsetlin patterns, the Γ -array and the weight associated to a pattern from [BBF11b, Chapter 2].

Definition 2. A *Gelfand-Tsetlin pattern* of rank r and top row λ is an array of nonnegative integers

$$(4.1) \quad \mathfrak{T} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0,r-1} & a_{0r} \\ & a_{11} & & & & a_{1r} \\ & & \ddots & & & \\ & & & a_{rr} & & \end{pmatrix}$$

where the top row is $\lambda = (a_{00}, a_{01}, \dots, a_{0,r-1}, a_{0,r}) = (\lambda_1, \lambda_2, \dots, \lambda_r, \lambda_{r+1})$, and rows are non-increasing and interleave: $a_{i-1,j-1} \geq a_{ij} \geq a_{i-1,j}$.

For every $1 \leq i \leq r$, let

$$(4.2) \quad \Gamma_{ij} = \Gamma_{ij}(\mathfrak{T}) = \sum_{k=j}^r (a_{i,k} - a_{i-1,k}).$$

This gives the Γ -array of \mathfrak{T}

$$(4.3) \quad \Gamma(\mathfrak{T}) = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1r} \\ & \Gamma_{22} & \cdots & \Gamma_{2r} \\ & & \ddots & \vdots \\ & & & \Gamma_{1r} \end{bmatrix}.$$

Remark 2. Note that given the top row, the entries of the Gelfand-Tsetlin pattern \mathfrak{T} can be recovered from the entries of $\Gamma(\mathfrak{T})$. That is, given $a_{0,i}$ and $\Gamma_{i,j}$ for $1 \leq i \leq j \leq r$, one can compute each $a_{i,j}$.

Since the entries of the Gelfand-Tsetlin pattern \mathfrak{T} satisfy $0 \leq a_{i,k} - a_{i-1,k} \leq a_{i-1,k-1} - a_{i-1,k}$, we have

$$(4.4) \quad 0 \leq \Gamma_{ir} \leq a_{i-1,r-1} - a_{i-1,r}; \quad \forall i \leq l \leq r-1 \quad \Gamma_{i,l+1} \leq \Gamma_{i,l} \leq \Gamma_{i,l+1} + a_{i-1,l-1} - a_{i-1,l};$$

so the rows in $\Gamma(\mathfrak{T})$ are nonnegative, non-increasing and there is an upper bound on the difference of consecutive entries in a row. The Gelfand-Tsetlin coefficient assigned to \mathfrak{T} depends on the *decoration* of \mathfrak{T} , i.e. whether these inequalities are strict or not. We recall the relevant terminology here.

Definition 3. (*Decorations of the entries of $\Gamma(\mathfrak{T})$ and \mathfrak{T} .*) An entry of $\Gamma(\mathfrak{T})$ may be *undecorated*, *circled*, *boxed*, or *both*. The table below shows the (“right-leaning”) rules for decorating $\Gamma(\mathfrak{T})$. (If $j = r$, take $\Gamma_{i,r+1} = 0$.)

$$(4.5) \quad \begin{array}{c|c} \Gamma_{i,j+1} = \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} & \Gamma_{ij} \text{ is circled} \\ \hline \Gamma_{i,j+1} < \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} & \Gamma_{ij} \text{ is undecorated} \\ \hline \Gamma_{i,j+1} < \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} & \Gamma_{ij} \text{ is boxed} \\ \hline \Gamma_{i,j+1} = \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} & \Gamma_{ij} \text{ is circled and boxed} \end{array}$$

We may phrase this as decorating the entries (below the top row) of the Gelfand-Tsetlin pattern \mathfrak{T} itself. The decoration of $a_{i,j}$ is the same as that of $\Gamma_{i,j}$.

$$(4.6) \quad \begin{array}{c|c} a_{i-1,j} = a_{ij} < a_{i-1,j-1} & a_{ij} \text{ is circled} \\ \hline a_{i-1,j} < a_{ij} < a_{i-1,j-1} & a_{ij} \text{ is undecorated} \\ \hline a_{i-1,j} < a_{ij} = a_{i-1,j-1} & a_{ij} \text{ is boxed} \\ \hline a_{i-1,j} = a_{ij} = a_{i-1,j-1} & a_{ij} \text{ is circled and boxed} \end{array}$$

Let d_i denote the sum of the entries in the i -th row of \mathfrak{T} , that is,

$$(4.7) \quad d_i = d_i(\mathfrak{T}) = \sum_{j=i}^r a_{ij}.$$

Then we may define the *weight* of a Gelfand-Tsetlin pattern \mathfrak{T} .

$$(4.8) \quad \text{wt}(\mathfrak{T}) := (d_r, d_{r-1} - d_r, \dots, d_0 - d_1)$$

We conclude by an example.

Example 2. Consider Gelfand-Tsetlin patterns of top row $(3, 1, 0)$. One example of these is

$$\mathfrak{T} = \begin{pmatrix} 3 & 1 & 0 \\ & 3 & 1 \\ & & 2 \end{pmatrix}.$$

The corresponding Γ -array is

$$(4.9) \quad \Gamma(\mathfrak{T}) = \begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}.$$

The sums of elements in the rows of the pattern \mathfrak{T} are $d_0 = 4$, $d_1 = 4$ and $d_2 = 2$, hence

$$\text{wt}(\mathfrak{T}) = (d_2, d_1 - d_2, d_0 - d_1) = (2, 2, 0).$$

4.3. Berenstein-Zelevinsky-Littelmann paths. To a vertex v in the crystal \mathcal{C}_λ and a choice of reduced decomposition for the long element $w_0 \in W$ corresponds a Berenstein-Zelevinsky-Littelmann path. This is a path in the graph theoretic sense. It starts from v , steps along the directed edges of the crystal, and ends in the lowest element, v_{lowest} . The steps correspond to applying successive Kashiwara operators f_i to v . The direction of steps is dictated by the choice of a long word w_0 . The notation follows [BBF11b]; an explicit type A_2 example is included after the definition.

4.3.1. *Choice of the long word.* Let

$$(4.10) \quad w_0 = \sigma_1 \sigma_2 \sigma_1 \cdots \sigma_{r-1} \cdots \sigma_1 \sigma_r \cdots \sigma_1.$$

This is our reduced expression of choice for the longest element in S_{r+1} (our “favourite long word”). Let $1 \leq \Omega_i \leq r$ ($1 \leq i \leq N = \ell(w_0)$) be the indices so that

$$(4.11) \quad w_0 = \sigma_{\Omega_1} \sigma_{\Omega_2} \cdots \sigma_{\Omega_N},$$

is the same reduced expression as in (4.10), i.e. $\Omega_1 = 1, \Omega_2 = 2, \Omega_3 = 1, \dots, \Omega_N = 1$.

4.3.2. *Building the path.* Let v be any element of the highest weight crystal \mathcal{C}_λ . Recall that for any vertex $w \in \mathcal{C}_\lambda$ and any $1 \leq i \leq r$, we may have either $f_i(w) \in \mathcal{C}_\lambda$, in which case $\text{wt}(f_i(w)) = \text{wt}(w) - \alpha_i$, or $f_i(w) = 0$. Let $b_1 := \varphi_1(v)$, i.e. let b_1 be a largest integer such that $f_{\Omega_1}^{b_1} v \neq 0$. Let $v_1 = f_{\Omega_1}^{b_1} v$, and similarly for $i = 2, \dots, N$ let b_i be the largest integer such that $(v_i :=) f_{\Omega_i}^{b_i} v_{i-1} \neq 0$ (i.e. $b_i := \varphi_{\Omega_i}(v_{i-1})$). We may write these integers into an array.

$$(4.12) \quad BZL(v) = BZL_\Omega(v) = \begin{bmatrix} b_{\binom{r}{2}+1} & b_{\binom{r}{2}+2} & \cdots & b_{\binom{r+1}{2}} \\ & b_{\binom{r-1}{2}+1} & \cdots & b_{\binom{r}{2}} \\ & & \ddots & \\ & & & b_2 & b_3 \\ & & & & b_1 \end{bmatrix}$$

Example 3. Let $r = 2$, and $\lambda = (3, 1, 0)$. We have $w_0 = \sigma_1 \sigma_2 \sigma_1$. Let $v = v_{(2,2,0)}$ be the single vertex of $\mathcal{C}_{(3,1,0)}$ with $\text{wt}(v) = (2, 2, 0)$ (see Figure 1 and Figure 2). Then $b_1 = 1$, $b_2 = 3$ and $b_3 = 1$, and the BZL array of v is

$$(4.13) \quad BZL(v) = \begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}.$$

Notice that this is the same as the Γ -array in (4.9). The BZL path corresponding to v is as shown on Figure 3.

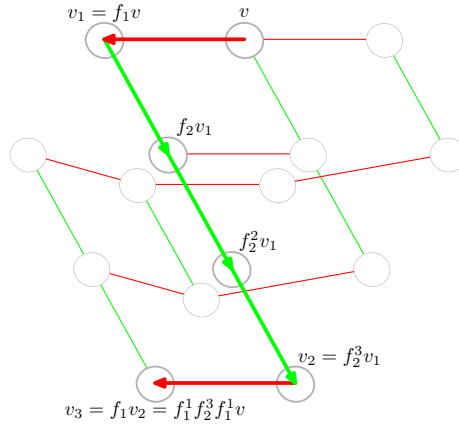


FIGURE 3. The Berenstein-Zelevinsky-Littelmann path of $v_{(2,2,0)} \in \mathcal{C}_{(3,1,0)}$.

4.3.3. Correspondence of crystals and patterns.

Proposition 10. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \lambda_{r+1})$ be a dominant weight, \mathcal{C}_λ the crystal with highest weight λ .*

- (i) *For any $v \in \mathcal{C}_\lambda$ the BZL-path of v “ends” in the lowest element $v_{\text{lowest}} \in \mathcal{C}_\lambda$, i.e. $v_{\binom{r+1}{2}} = v_{\text{lowest}}$.*
- (ii) *A vertex v can be recovered from $BZL(v)$.*
- (iii) *For any $v \in \mathcal{C}_\lambda$ and $BZL(v) = (b_i)_{1 \leq i \leq \binom{r+1}{2}}$ as above, we have*

$$(4.14) \quad \text{wt}(v) - \text{wt}(v_{\text{lowest}}) = \sum_{i=1}^{\binom{r+1}{2}} b_i \cdot \alpha_{\Omega_i}.$$

- (iv) *Elements of the crystal \mathcal{C}_λ are in bijection with Gelfand-Tsetlin patterns with top row λ . The correspondence is given by assigning $\mathfrak{T}(v)$ to v if and only if $BZL(v) = \Gamma(\mathfrak{T})$.*
- (v) *With the correspondence as above, we have $\text{wt}(v) = \text{wt}(\mathfrak{T}(v))$.*

Proof. Parts of this proposition are proved throughout Chapter 2 of [BBF11b]. In particular, [BBF11b, Lemma 2.1] proves (i) and (ii); [BBF11b, Proposition 2.3] proves (iii) and (v). The correspondence in (iv) is proved using Young-tableaux. Some of the relevant proofs in [BBF11b] use Berenstein and Zelevinsky [BZ93, BZ⁺96], Kirillov and Berenstein [KB96], Littelmann [Lit98] and Lusztig [Lus90] as a reference. \square

4.4. Gelfand-Tsetlin coefficients. In this section, we define the coefficients appearing on the right-hand side of Theorem 1. The definitions depend on a positive integer n (the degree of the metaplectic cover), the corresponding Gauss sums $g^b(a)$ and $h^b(a)$ defined in section 3.3, and the decorations of arrays introduced in Definition 3.

By remark 2, a pattern \mathfrak{T} can be recovered from $\Gamma(\mathfrak{T})$ and the top row λ . Since many computations in the sequel involve a fixed n and λ , we often suppress these from the notation. We write $G^{(n, \lambda)}(\mathfrak{T}) = G^{(n)}(\mathfrak{T}) = G(\mathfrak{T})$ when \mathfrak{T} is understood to be a pattern with top row λ . We write $G(\mathfrak{T}) = G^{(n, \lambda)}(\Gamma) = G^{(\lambda)}(\Gamma) = G(\Gamma)$ when $\Gamma = \Gamma(\mathfrak{T})$, and $G^{(n, \lambda)}(v) = G^{(\lambda)}(v) = G(v)$ when $v \in \mathcal{C}_\lambda$ corresponds to \mathfrak{T} by Proposition 10.

Definition 4. Let \mathfrak{T} be a Gelfand-Tsetlin pattern with top row λ , $\Gamma(\mathfrak{T}) = (\Gamma_{ij})_{1 \leq i \leq j \leq r}$ its Γ -array as in (4.3). Then the degree n Gelfand-Tsetlin coefficient corresponding to \mathfrak{T} is

$$(4.15) \quad G^{(n)}(\mathfrak{T}) = \prod_{1 \leq i \leq j \leq r} g_{ij}^n(\mathfrak{T}),$$

where $g_{ij}(\mathfrak{T}) = g_{ij}^{(n)}(\mathfrak{T})$ is given below.

$$(4.16) \quad g_{ij}(\mathfrak{T}) = \begin{cases} 1 & \Gamma_{i,j+1} = \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j}, \quad \text{i.e. } a_{ij} \text{ is undecorated} \\ h^b(\Gamma_{ij}) & \Gamma_{i,j+1} < \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j}, \quad \text{i.e. } a_{ij} \text{ is circled} \\ g^b(\Gamma_{ij}) & \Gamma_{i,j+1} < \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j}, \quad \text{i.e. } a_{ij} \text{ is boxed} \\ 0 & \Gamma_{i,j+1} = \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j}, \quad \text{i.e. } a_{ij} \text{ is circled and boxed} \end{cases}$$

The coefficient depends strongly on n . To elucidate this, we give the examples of the nonmetaplectic case ($n = 1$) and the simplest metaplectic case ($n = 2$) explicitly below. Recall from section 3.3 that $t^n = v = q^{-1}$, where q is the cardinality of a residue field $\mathfrak{o}_S/p\mathfrak{o}_S$.

Example 4. When $n = 1$, the factors $g_{ij}^{(n)}(\mathfrak{T})$ of the Gelfand-Tsetlin coefficient $G^{(n)}(\mathfrak{T})$ are as follows.

$$(4.17) \quad g_{ij}^{(1)}(\Gamma) = \begin{cases} 1 & \Gamma_{i,j+1} = \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \\ 1-t & \Gamma_{i,j+1} < \Gamma_{ij} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \\ -t & \Gamma_{i,j+1} < \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \\ 0 & \Gamma_{i,j+1} = \Gamma_{ij} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} \end{cases}$$

Let us compute the Gelfand-Tsetlin coefficient of the pattern in Example 2. Recall that this pattern corresponds to the single element of $\mathcal{C}_{(3,1,0)}$ of weight $(2, 2, 0)$.

$$\mathfrak{T}(v_{(2,2,0)}) = \begin{pmatrix} 3 & 1 & 0 \\ & 3 & 1 \\ & & 2 \end{pmatrix}, \quad \text{and} \quad \Gamma(\mathfrak{T}(v_{(2,2,0)})) = \begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}.$$

Here a_{11} and a_{12} (or Γ_{11} and Γ_{12}) are boxed, while a_{22} (or Γ_{22}) is undecorated. Thus we have $G^{(1)}(\mathfrak{T}(v_{(2,2,0)})) = (-t)^2 \cdot (1-t)$.

Example 5. Let $n = 2$. Then the factors $g_{ij}^{(n)}(\mathfrak{T})$ of the Gelfand-Tsetlin coefficient $G^{(n)}(\mathfrak{T}) = G^{(n)}(\Gamma)$ are

$$(4.18) \quad g_{ij}^{(2)}(\Gamma) = \begin{cases} 1 & \Gamma_{i,j+1} = \Gamma_{i,j} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j} & \Gamma_{ij} \text{ is circled} \\ 1-t^2 & \Gamma_{i,j+1} < \Gamma_{i,j} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j}; 2 \mid \Gamma_{i,j} & \Gamma_{ij} \text{ is undecorated} \\ 0 & \Gamma_{i,j+1} < \Gamma_{i,j} < \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j}; 2 \nmid \Gamma_{i,j} & \Gamma_{ij} \text{ is undecorated} \\ -t^2 & \Gamma_{i,j+1} < \Gamma_{i,j} = \Gamma_{i,j+1} + a_{i-1,j-1} - a_{i-1,j}; 2 \mid \Gamma_{i,j} & \Gamma_{ij} \text{ is boxed} \\ t & \Gamma_{i,j+1} < \Gamma_{i,j} = \Gamma_{i,j+1} + \lambda_j - \lambda_{j+1} + 1; 2 \nmid \Gamma_{i,j} & \Gamma_{ij} \text{ is boxed} \\ 0 & \Gamma_{i,j+1} = \Gamma_{i,j} = \Gamma_{i,j+1} + \lambda_j - \lambda_{j+1} + 1; & \Gamma_{ij} \text{ is circled, boxed} \end{cases}$$

Notice that the factors depend on the residue of Γ_{ij} modulo $n = 2$. Returning to the example of $v_{(2,2,0)} \in \mathcal{C}_{(3,1,0)}$, we see that since $\Gamma_{22} = 1$ is undecorated and odd, $G^{(2)}(\mathfrak{T}(v_{(2,2,0)})) = t^2 \cdot 0 = 0$.

4.5. Branching properties. The following branching rule of type A_r highest-weight crystals is well known. (See, for example, [BBF11b, (2.4)].) We shall adapt it to the metaplectic setting, and Demazure crystals in section 6; these adapted branching rules play a key role in the proof of Theorem 1.

Proposition 11. *When all the edges of a highest weight crystal $\mathcal{C}_{\lambda+\rho}$ labelled by r are removed, the connected components of the result are all isomorphic to highest weight crystals \mathcal{C}_μ of type A_{r-1} . Omitting the last component of $\text{wt} : \mathcal{C}_{\lambda+\rho} \rightarrow \mathbb{Z}^{r+1}$, and restricting it to a connected component gives the weight function on that component:*

$$(4.19) \quad \text{wt}_\mu : \mathcal{C}_\mu \rightarrow \mathbb{Z}^r.$$

The highest weights μ that appear in this decomposition are dominant and interleave with $\lambda + \rho$. We identify the highest weight crystal \mathcal{C}_μ with the appropriate subcrystal of $\mathcal{C}_{\lambda+\rho}$. That is, we have

$$(4.20) \quad \mathcal{C}_{\lambda+\rho} = \bigcup_{\mu} \mathcal{C}_\mu.$$

and the (disjoint) union is over all $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ such that

$$(4.21) \quad \lambda_1 + r \geq \mu_1 \geq \lambda_2 + r - 1 \geq \dots \geq \lambda_r + 1 \geq \mu_r \geq \lambda_{r+1}.$$

An element $v \in \mathcal{C}_{\lambda+\rho}$ belongs to \mathcal{C}_μ in the disjoint union (4.20) if the second row of the pattern $\mathfrak{T}(v)$ is $(a_{11}, a_{12}, \dots, a_{1r}) = \mu$. (Here $\mathfrak{T}(v)$ is the Gelfand-Tsetlin pattern with top row $\lambda + \rho$ corresponding to v as in Proposition 10.)

Example 6. If $\lambda + \rho = (3, 1, 0)$, then the weights μ are $(3, 1)$, $(3, 0)$, $(2, 1)$, $(2, 0)$, $(1, 1)$ and $(1, 0)$. Figure 4 shows the corresponding components of $\mathcal{C}_{(3,1,0)}$. These are of Cartan type A_1 . The highest element of each string is labeled with the corresponding weight μ .

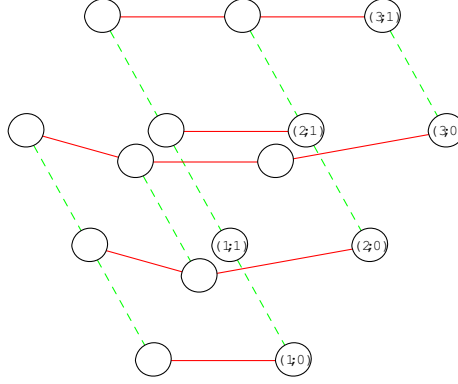


FIGURE 4. The A_1 components (1-strings) of $\mathcal{C}_{(3,1,0)}$.

The remainder of this section is dedicated to describing the weights and Gelfand-Tsetlin coefficients on components in (4.20) explicitly. The results are summarized in Proposition 12 below. As before, we use the notation $\mathbf{x} = (x_1, \dots, x_r, x_{r+1})$, write $\mathbf{y} = (x_1, \dots, x_r)$ and let $d(\lambda)$ (or $d(\mu)$) denote the sum of the components of the weight λ (respectively, μ).

Proposition 12. Let \mathcal{C}_μ be one of the components in the decomposition (4.20) of $\mathcal{C}_{\lambda+\rho}$, i.e. suppose μ and $\lambda + \rho$ interleave. Let v be any element of \mathcal{C}_μ . Then we have the following.

(a) If $\text{wt}_\mu : \mathcal{C}_\mu \rightarrow \mathbb{Z}^r$ denotes the weight function on \mathcal{C}_μ , then

$$(4.22) \quad \mathbf{x}^{\text{wt}(v)} = \mathbf{y}^{\text{wt}_\mu(v)} \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)}.$$

(b) Let v_* denote the lowest element of \mathcal{C}_μ (as a type A_{r-1} crystal).

$$(4.23) \quad G^{(n,\lambda+\rho)}(v) = G^{(n,\mu)}(v) \cdot G^{(n,\lambda+\rho)}(v_*).$$

Proof. Let $\mathfrak{T}(v) = \mathfrak{T}_{\lambda+\rho}(v)$ be the Gelfand-Tsetlin pattern corresponding to $v \in \mathcal{C}_{\lambda+\rho}$ as in Proposition 10. By Proposition 11, as an element of \mathcal{C}_μ , v corresponds to the pattern $\mathfrak{T}_\mu(v)$, and $\mathfrak{T}_\mu(v)$ is the same as $\mathfrak{T}_{\lambda+\rho}(v)$ with its first row omitted. In particular, the second row of $\mathfrak{T}(v)$ is μ . Thus by (4.7), $d_0(\mathfrak{T}(v)) = d(\lambda)$ and $d_1(\mathfrak{T}(v)) = d(\mu)$. By (4.8), the last coordinate of $\text{wt}(\mathfrak{T}(v))$ is $d(\lambda) - d(\mu)$. This implies (a). For (b), we restrict our attention to weights μ that are strongly dominant, i.e. we would like to assume $\mu_1 > \mu_2 > \dots > \mu_r$. We can do this because of the following remark.

Remark 3. The statement (4.23) is trivial if μ is not strongly dominant. By Remark 5, the Gelfand-Tsetlin coefficient of a non-strict pattern is zero. The second row of $\mathfrak{T}(v)$ is μ , hence if μ is not strongly dominant, then $\mathfrak{T}(v)$ is non-strict for any $v \in \mathcal{C}_\mu$, and $G^{(n,\lambda+\rho)}(v) = G^{(n,\lambda+\rho)}(v_*) = 0$.

Assume that μ is strongly dominant, hence the first two rows of $\mathfrak{T}(v)$ are strict for every $v \in \mathcal{C}_{\lambda+\rho}$. The next remark describes the *BZL* path of elements in \mathcal{C}_μ .

Remark 4. Recall from section 4.3 that the *BZL* path of v is a path in $\mathcal{C}_{\lambda+\rho}$ from v to $v_{\text{lowest}} \in \mathcal{C}_{\lambda+\rho}$. The j th segment of the path is along an edge of $\mathcal{C}_{\lambda+\rho}$ labelled by Ω_j , where Ω_j is defined in (4.10). The chosen long word $w_0^{(r)}$ starts with $w_0^{(r-1)}$; in particular the first $\binom{r}{2}$ out of the $\binom{r+1}{2}$ segments are along edges *not* labelled by r . This implies that these segments are contained in \mathcal{C}_μ , and in fact are the *BZL* path corresponding to v as an element of \mathcal{C}_μ , a crystal of type A_{r-1} . Hence the end of the first $\binom{r}{2}$ segments is the lowest element of that crystal, v_* . Consequently, the first $\binom{r}{2}$ segments of the *BZL* path of v_* are trivial. If $b_j(v) = b_j(\mathfrak{T}(v))$ denotes the length of the j th segment of the *BZL* path, then

$$(4.24) \quad b_j(v_*) = 0 \text{ for } j \leq \binom{r}{2}, \text{ and } b_j(v) = b_j(v_*) \text{ for every } v \in \mathcal{C}_\mu \text{ and } j > \binom{r}{2}.$$

Thus $BZL(v_*) = \Gamma(\mathfrak{T}(v_*))$ has zeros everywhere below the first row. By (4.2), this means that for any $1 < i \leq j \leq r+1$, the entry $a_{i,j}$ of $\mathfrak{T}(v_*)$ satisfies $a_{i,j} = a_{1,j} = \mu_j$. It follows that we have $a_{i-1,j-1} > a_{i,j} = a_{i-1,j}$. According to Definition 3, these entries are all circled, but not boxed. By Definition 4 this implies

$$(4.25) \quad G^{(n,\lambda+\rho)}(\mathfrak{T}(v_*)) = \prod_{1 \leq i \leq j \leq r} g_{ij}^{n,\lambda+\rho}(\mathfrak{T}(v_*)) = \prod_{1 \leq j \leq r} g_{1j}^{n,\lambda+\rho}(\mathfrak{T}(v_*)) \cdot \prod_{2 \leq i \leq j \leq r} 1.$$

The first two rows of $\mathfrak{T}(v)$ are $\lambda + \rho$ and μ for every $v \in \mathcal{C}_\mu$. The coefficient $g_{1j}^{n,\lambda+\rho}(\mathfrak{T}(v))$ only depends on those two rows; hence its value is the same for any $v \in \mathcal{C}_\mu$ and v_* . Hence we have

$$(4.26) \quad G^{(n,\lambda+\rho)}(\mathfrak{T}(v)) = \prod_{1 \leq i \leq j \leq r} g_{ij}^{n,\lambda+\rho}(\mathfrak{T}(v)) = \prod_{1 \leq j \leq r} g_{1j}^{n,\lambda+\rho}(\mathfrak{T}(v_*)) \cdot \prod_{2 \leq i \leq j \leq r} g_{ij}^{n,\lambda+\rho}(\mathfrak{T}(v))$$

The first product here is equal to $G^{(n,\lambda+\rho)}(\mathfrak{T}(v_*))$ by (4.25). The second factor,

$$(4.27) \quad \prod_{2 \leq i \leq j \leq r} g_{ij}^{n,\lambda+\rho}(\mathfrak{T}(v)) = G^{(n,\mu)}(v),$$

because, as seen above, the Gelfand-Tsetlin pattern $\mathfrak{T}_\mu(v)$ corresponding to v as an element of \mathcal{C}_μ is $\mathfrak{T}(v) = \mathfrak{T}_{\lambda+\rho}(v)$ minus its first row. Thus, substituting (4.25) and (4.27) into (4.26) gives (4.23). \square

5. TOKUYAMA'S THEOREM

Tokuyama's theorem, in its original form, relates a Schur function to a generating function of strict Gelfand patterns. This is easily rephrased to relate a sum over a Weyl group to a sum over a highest weight crystal. This second form is more convenient for the purposes of generalizing the theorem to the metaplectic setting.

In the previous section, we followed notation from Brubaker-Bump-Friedberg [BBF11b], because that is most convenient to use for metaplectic definitions of Gelfand-Tsetlin coefficients. The notation and approach in Tokuyama's paper [Tok88] is slightly different. Here we phrase Tokuyama's theorem using both sets of notation, and explain why the two versions are equivalent.

Let $\mathbf{x} = (x_1, \dots, x_{r+1})$, $\mathbf{z} = (z_1, \dots, z_{r+1})$, $\lambda = (\lambda_1, \dots, \lambda_{r+1})$ and let $\rho = (r, r-1, \dots, 1, 0)$ be the Weyl vector. Let $s_\lambda(\mathbf{x})$ (or $s_\lambda(\mathbf{z})$) denote the Schur function associated to the highest-weight representation of GL_{r+1} with highest weight λ . Recall that a Gelfand-Tsetlin pattern is an array of the form (4.1), where rows are non-increasing and interleave.

As in Tokuyama [Tok88], we say a pattern \mathfrak{T} is strict if $a_{i-1,j-1} > a_{i-1,j}$ holds for every $1 \leq i \leq j \leq r$. Following notation there, let $G(\lambda)$ denote the set of Gelfand-Tsetlin patterns with top row λ , and let $\mathrm{SG}(\lambda)$ be the set of strict Gelfand-Tsetlin patterns with top row λ .

Remark 5. Note that by Definition 3, a Gelfand-Tsetlin pattern \mathfrak{T} is strict if and only if it has no entries that are both circled and boxed. In every version of Gelfand-Tsetlin coefficients, such an entry corresponds to a factor of zero. Hence as long as each term of the sum involves the Gelfand-Tsetlin coefficients, summing over $G(\lambda)$ is the same as summing over $\mathrm{SG}(\lambda)$.

Recall that if d_i is the sum of elements in the i -th row (4.7), then by (4.8) the weight of a pattern \mathfrak{T} is $\mathrm{wt}(\mathfrak{T}) = (d_r, d_{r-1} - d_r, \dots, d_0 - d_1)$. In Tokuyama [Tok88], we have

$$(5.1) \quad M(\mathfrak{T}) = (d_0 - d_1, d_1 - d_2, \dots, d_{r-1} - d_r, d_r).$$

For a weight $\mu = (\mu_1, \mu_2, \dots, \mu_{r+1})$ write $\mathbf{x}^\mu = x_1^{\mu_1} \cdot x_2^{\mu_2} \cdots x_{r+1}^{\mu_{r+1}}$. Recall the definition of the (nonmetaplectic) Gelfand-Tsetlin coefficient $G(\mathfrak{T}) = G^{(1)}(\mathfrak{T})$ as a product of $g_{ij}(\mathfrak{T})$ from (4.15) and (4.17). Let us treat t as an indeterminate for the time being. Then the factor $g_{ij}(\mathfrak{T})$ corresponding to an entry a_{ij} is as follows.

$$g_{ij}(\mathfrak{T}) = \begin{cases} 1 & a_{i-1,j} = a_{ij} & a_{ij} \text{ is circled} \\ 1-t & a_{i-1,j} < a_{ij} < a_{i-1,j-1} & a_{ij} \text{ is undecorated} \\ -t & a_{i-1,j} < a_{ij} = a_{i-1,j-1} & a_{ij} \text{ is boxed.} \\ 0 & a_{i-1,j} = a_{ij} = a_{i-1,j-1} & a_{ij} \text{ is circled and boxed.} \end{cases}$$

In Tokuyama [Tok88], the entry a_{ij} is called “special” if $a_{i-1,j} < a_{ij} < a_{i-1,j-1}$ and “lefty” if $a_{i-1,j} = a_{ij}$. By Definition 3 “special” entries are undecorated, and “lefty” entries are boxed. (For strict patterns, “lefty” entries are boxed and not circled by Remark 5.)

We are now ready to state Tokuyama’s theorem in both the notation of [Tok88] and in ours.

Theorem 13. (*Tokuyama’s theorem*) Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \lambda_{r+1}) \in \mathbb{Z}^{r+1}$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r+1} \geq 0$, and $\rho = (r, r-1, \dots, 1, 0)$, $\mathrm{SG}(\lambda + \rho)$, and $M(\mathfrak{T})$ as defined above. Let $s(\mathfrak{T})$ be the number of special entries of \mathfrak{T} and $l(\mathfrak{T})$ the number of lefty entries.

$$(5.2) \quad s_\lambda(\mathbf{z}) \cdot \prod_{1 \leq i < j < r+1} (z_i - t \cdot z_j) = \sum_{\mathfrak{T} \in \mathrm{SG}(\lambda + \rho)} (1-t)^{s(\mathfrak{T})} \cdot (-t)^{l(\mathfrak{T})} \cdot \mathbf{z}^{M(\mathfrak{T})}.$$

In the notation introduced in previous sections of this chapter, this can be re-written as

$$(5.3) \quad s_\lambda(\mathbf{x}) \cdot \prod_{1 \leq i < j < r+1} (x_j - t \cdot x_i) = \sum_{\mathfrak{T} \in G(\lambda + \rho)} G(\mathfrak{T}) \cdot \mathbf{x}^{\mathrm{wt}(\mathfrak{T})}.$$

The first form of the equation, (5.2) is Theorem 2.1 of [Tok88], substituting $-t$ for t . We explain why (5.3) is equivalent. Note that (thinking of t as an indeterminate) for any strict Gelfand-Tsetlin pattern we have $G(\mathfrak{T}) = (1-t)^{s(\mathfrak{T})} \cdot (-t)^{l(\mathfrak{T})}$. Furthermore, by Remark 5 we have $G(\mathfrak{T}) = 0$ if $\mathfrak{T} \in G(\lambda + \rho) \setminus \mathrm{SG}(\lambda + \rho)$. From (4.8) and (5.1) we see that the components of $\mathrm{wt}(\mathfrak{T})$ are exactly the components of $M(\mathfrak{T})$ in reverse order. So if we write $x_1 = z_{r+1}$, $x_2 = z_r$, \dots , $x_r = z_2$, $x_{r+1} = z_1$, we have $\mathbf{x}^{\mathrm{wt}(\mathfrak{T})} = \mathbf{z}^{M(\mathfrak{T})}$. Hence the right hand sides of (5.2) and (5.3) agree. It remains to check that the left hand sides agree as well.

Note that with the choice $x_i = z_{r+2-i}$, we have $s_\lambda(\mathbf{x}) = s_\lambda(\mathbf{z})$ and $\prod_{1 \leq i < j < r+1} (z_i - t \cdot z_j) = \prod_{1 \leq i < j < r+1} (x_j - t \cdot x_i)$.

In the remainder of this section, we reformulate Theorem 13 in terms of Demazure-Lusztig operators and a sum over a highest-weight crystal. This is done separately for the two sides.

5.1. The right hand side of Tokuyama's theorem as a sum over a crystal. The correspondence between elements of a crystal of highest weight $\lambda + \rho$ and Gelfand-Tsetlin patterns of top row $\lambda + \rho$ was established by Proposition 10. Following that parametrization and notation, we may write $G(v) = G^{(1)}(v) := G^{(1)}(\mathfrak{T}(v))$; we have $\text{wt}(\mathfrak{T}(v)) = \text{wt}(v)$. Thus we may write the right hand side of (5.3) as

$$(5.4) \quad \sum_{\mathfrak{T} \in G(\lambda + \rho)} G(\mathfrak{T}) \cdot \mathbf{x}^{\text{wt}(\mathfrak{T})} = \sum_{v \in \mathcal{C}_{\lambda + \rho}} G(v) \cdot \mathbf{x}^{\text{wt}(v)}.$$

5.2. The left hand side of Tokuyama's theorem in terms of Demazure-Lusztig operators. Recall notation for type A root systems in Section 3.1. This notation and the Weyl Character Formula for Schur functions allows us to rewrite the left hand side of (5.3) first as a sum over the Weyl group. Then we use Theorems 6 and 7 to write it in terms of Demazure-Lusztig operators.

The Weyl group $W \cong S_{r+1}$ acts on $\Lambda = \mathbb{Z}^{r+1}$ by permuting the coordinates. Thus for the long element w_0 we have

$$(5.5) \quad \prod_{1 \leq i < j < r+1} (x_j - t \cdot x_i) = \mathbf{x}^{w_0 \cdot \rho} \cdot \prod_{\alpha \in \Phi^+} (1 - t \cdot \mathbf{x}^\alpha).$$

Now by the Weyl Character Formula we have

$$s_\lambda(\mathbf{x}) = \frac{\sum_{w \in W} \text{sgn}(w) \cdot w \cdot (\mathbf{x}^{\lambda + \rho})}{\prod_{1 \leq i < j < r+1} (x_i - x_j)} = \frac{\sum_{w \in W} \text{sgn}(w) \cdot w \cdot (\mathbf{x}^{\lambda + \rho})}{\mathbf{x}^{w_0 \cdot \rho} \cdot \text{sgn}(w_0) \cdot \prod_{\alpha \in \Phi^+} (1 - \mathbf{x}^\alpha)}.$$

Thus the left hand side of (5.3) can be rewritten as

$$s_\lambda(\mathbf{x}) \cdot \prod_{1 \leq i < j < r+1} (x_j - t \cdot x_i) = \frac{\prod_{\alpha \in \Phi^+} (1 - t \cdot \mathbf{x}^\alpha)}{\prod_{\alpha \in \Phi^+} (1 - \mathbf{x}^\alpha)} \cdot \text{sgn}(w_0) \cdot \sum_{w \in W} \text{sgn}(w) \cdot w \cdot (\mathbf{x}^{\lambda + \rho}).$$

Recall the notation Δ_t defined in (3.20) for a deformation the Weyl denominator; write Δ for Δ_t when $t = 1$. Then we have

$$(5.6) \quad s_\lambda(\mathbf{x}) \cdot \prod_{1 \leq i < j < r+1} (x_j - t \cdot x_i) = \frac{\Delta_t}{\Delta} \cdot \text{sgn}(w_0) \cdot \sum_{w \in W} \text{sgn}(w) \cdot w \cdot (\mathbf{x}^{\lambda + \rho}).$$

Substituting ww_0 for w we may write (5.6) as

$$s_\lambda(\mathbf{x}) \cdot \prod_{1 \leq i < j < r+1} (x_j - t \cdot x_i) = \frac{\Delta_t}{\Delta} \cdot \sum_{w \in W} \text{sgn}(w) \cdot w \cdot (\mathbf{x}^{w_0(\lambda + \rho)})$$

Now, to write this as a linear combination of Weyl group elements acting on $\mathbf{x}^{w_0 \lambda}$, notice that as operators, $w \cdot \mathbf{x}^{w_0 \rho} = (w \cdot \mathbf{x}^{w_0 \rho}) \cdot w = \mathbf{x}^{w_0 \rho} \cdot w$. With $\Phi(w)$ as in (3.1) we have $\Phi(w^{-1}) = w(\Phi^-) \cap \Phi^+ = ww_0(\Phi^+) \cap \Phi^+$, and hence $ww_0\rho - w_0\rho = \sum_{\alpha \in \Phi(w^{-1})} \alpha$, whence as operators, $w \cdot \mathbf{x}^{w_0 \rho} = \mathbf{x}^{w_0 \rho} \cdot \prod_{\alpha \in \Phi(w^{-1})} \mathbf{x}^\alpha \cdot w$. Thus we may rewrite (5.6) as

$$s_\lambda(\mathbf{x}) \cdot \prod_{1 \leq i < j < r+1} (x_j - t \cdot x_i) = \mathbf{x}^{w_0 \rho} \cdot \frac{\Delta_t}{\Delta} \cdot \sum_{w \in W} \left(\text{sgn}(w) \cdot \prod_{\alpha \in \Phi(w^{-1})} \mathbf{x}^\alpha \right) \cdot w \cdot (\mathbf{x}^{w_0 \lambda}),$$

and, combining the special cases of Theorem 6 and 7 for $n = 1$, as

$$(5.7) \quad s_\lambda(\mathbf{x}) \cdot \prod_{1 \leq i < j < r+1} (x_j - t \cdot x_i) = \mathbf{x}^{w_0 \rho} \cdot \Delta_t \cdot \mathcal{D}_{w_0}(\mathbf{x}^{w_0(\lambda)}) = \mathbf{x}^{w_0 \rho} \cdot \sum_{w \in W} \mathcal{T}_w(\mathbf{x}^{w_0(\lambda)}).$$

Remark 6. In this section, we used “the Weyl vector” as $\rho = (r, r-1, \dots, 1, 0)$. As a result, we have

$$\mathbf{x}^\rho = (x_1 \cdot x_2 \cdots x_{r+1})^r \cdot \prod_{\alpha \in \Phi^+} \mathbf{x}^{\frac{1}{2}\alpha}.$$

In the computations above, the factor $(x_1 \cdot x_2 \cdots x_{r+1})^r$ was never written out explicitly, but all of the equations hold as written. The factor $x_1 \cdot x_2 \cdots x_{r+1}$ is symmetric under W , so as an operator, it commutes with any element of the Weyl group.

5.3. The crystal version of Tokuyama’s theorem. The following “crystal version” of Tokuyama’s theorem is a direct consequence of (5.3), (5.7) and (5.4).

Lemma 14. *Let λ be a dominant, effective weight. Then we have*

$$(5.8) \quad \sum_{w \in W} \mathcal{T}_w(\mathbf{x}^{w_0(\lambda)}) = \mathbf{x}^{-w_0 \rho} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}} G(v) \cdot \mathbf{x}^{\text{wt}(v)}.$$

Lemma 14 is the form of Tokuyama’s theorem that is convenient to generalize, as we will see in Section 7.

6. DEMAZURE CRYSTALS AND BRANCHING PROPERTIES

The statement of Theorem 1 involves, on one side, a sum over a Demazure crystal. In this section we give the definition of Demazure subcrystals $\mathcal{C}_{\lambda+\rho}^{(w)}$ within a type A highest weight crystal $\mathcal{C}_{\lambda+\rho}$, for certain elements w of the Weyl group. In preparation for the proof of Theorem 1, we discuss how the branching properties of section 4.5 restrict to Demazure crystals (Proposition 15 and Proposition 16). Finally, we introduce some terminology that will allow for lighter notation in the proof of Theorem 1 in Section 7 and Section 8.

We start by specifying the set of Weyl group elements w that appear in the statement of Theorem 1.

Definition 5. Recall that in (4.10) we fixed the following long word:

$$w_0 = w_0^{(r)} = \sigma_1 \sigma_2 \sigma_1 \cdots \sigma_{r-1} \cdots \sigma_1 \sigma_r \cdots \sigma_1 = \sigma_{\Omega_1} \sigma_{\Omega_2} \cdots \sigma_{\Omega_{\binom{r+1}{2}}}$$

We say that the element w is a *beginning section of the long word* $w_0^{(r)}$ if

$$(6.1) \quad w = \sigma_{\Omega_1} \sigma_{\Omega_2} \cdots \sigma_{\Omega_l} \text{ for some } l (= \ell(w)) \leq \binom{r+1}{2}.$$

Sometimes it is convenient to assume that w is a beginning section of $w_0^{(r)}$, but not of $w_0^{(r-1)}$. In this case w is of the form

$$(6.2) \quad w = w_0^{(r-1)} \sigma_r \cdots \sigma_{r-k}.$$

For such elements $w \in W$, we write $k := \ell(w) - \ell(w_0^{(r-1)}) - 1$.

Now we are ready to define Demazure crystals corresponding to a beginning section (6.1) of our favourite long word.

Definition 6. Let $\mathcal{C}_{\lambda+\rho}$ be a crystal of highest weight $\lambda + \rho$, and let w be a beginning section of the long word $w_0^{(r)}$, as in (6.1). Then the *Demazure crystal corresponding to w* is the crystal $\mathcal{C}_{\lambda+\rho}^{(w)}$ with vertices

$$(6.3) \quad \mathcal{C}_{\lambda+\rho}^{(w)} = \{v \in \mathcal{C}_{\lambda+\rho} \mid b_i(v) = 0 \text{ for all } i > \ell(w)\}.$$

Here $b_i(v)$ ($1 \leq i \leq \binom{r+1}{2}$) denotes the i -th entry of the Berenstein-Zelevinsky-Littelmann array $BZL(v)$ of an element $v \in \mathcal{C}_{\lambda+\rho}$.

To define a crystal structure $\mathcal{C}_{\lambda+\rho}^{(w)}$, we contend that as a directed graph it is a full subgraph of $\mathcal{C}_{\lambda+\rho}$. That is, the edges of $\mathcal{C}_{\lambda+\rho}^{(w)}$ are exactly the edges of $\mathcal{C}_{\lambda+\rho}$ with both ends in $\mathcal{C}_{\lambda+\rho}^{(w)}$.

Remark 7. Definition 6 means that an element $v \in \mathcal{C}_{\lambda+\rho}$ belongs to $\mathcal{C}_{\lambda+\rho}^{(w)}$ if and only if the BZL -path of v already reaches the lowest element of the crystal $\mathcal{C}_{\lambda+\rho}$ after the first $\ell(w)$ steps. With w a beginning segment of $w_0^{(r)}$ but not of $w_0^{(r-1)}$ as in (6.2), we have $\ell(w) = \binom{r}{2} + k + 1$, so $v \in \mathcal{C}_{\lambda+\rho}^{(w)}$ if and only if

$$(6.4) \quad BZL(v) = BZL_{\Omega}(v) = \begin{bmatrix} b_{\binom{r}{2}+1} & b_{\binom{r}{2}+2} & \cdots & b_{\binom{r}{2}+k+1} & 0 & \cdots & 0 \\ & b_{\binom{r-1}{2}+1} & & \cdots & & & b_{\binom{r}{2}} \\ & & \ddots & & & & \\ & & & & b_2 & b_3 \\ & & & & & b_1 \end{bmatrix}$$

i.e. $v \in \mathcal{C}_{\lambda+\rho}^{(w)}$ if and only if the last $\ell(w_0^{(r)}) - \ell(w) = r - k - 1$ segments of the BZL path of v are trivial.

The definition is illustrated by the following example, in type A_2 .

Example 7. Recall the crystal $\mathcal{C}_{3,1,0}$ of highest weight $(3,1,0)$ from Example 1. The Demazure subcrystal corresponding to $w = \sigma_1\sigma_2$ is the highlighted part of the crystal in Figure 5.

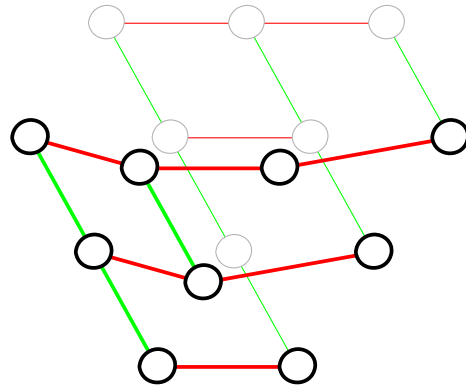


FIGURE 5. The Demazure crystal $\mathcal{C}_{(3,1,0)}^{(\sigma_1\sigma_2)}$ within $\mathcal{C}_{(3,1,0)}$.

For the remainder of this section, we assume that w is a beginning section of $w_0^{(r)}$, but not of $w_0^{(r-1)}$ i.e. it is as in (6.2). Then the Demazure crystal $\mathcal{C}_{\lambda+\rho}^{(w)}$ inherits some of the branching properties discussed in section 4.5, and in particular, Proposition 12. In particular, the type A_{r-1} subcrystals \mathcal{C}_μ from the decomposition (4.20) are either disjoint from, or contained in $\mathcal{C}_{\lambda+\rho}^{(w)}$. The set of μ such that \mathcal{C}_μ is contained in $\mathcal{C}_{\lambda+\rho}^{(w)}$ is easy to characterize from λ and w . We make this precise in the proposition below.

Proposition 15. *Let $w = w_0^{(r-1)} \cdots \sigma_r \cdots \sigma_{r-k}$ and $\mathcal{C}_{\lambda+\rho}^{(w)}$ the corresponding Demazure crystal. Let $\mathcal{C}_\mu \subset \mathcal{C}_{\lambda+\rho}$ be a subcrystal of type A_{r-1} from the decomposition (4.20). Then \mathcal{C}_μ has either no vertices in $\mathcal{C}_{\lambda+\rho}^{(w)}$, or it is contained in it. Furthermore, for a μ that interleaves with $\lambda + \rho$, we have $\mathcal{C}_\mu \subseteq \mathcal{C}_{\lambda+\rho}^{(w)}$ if and only if $\mu_j = \lambda_{j+1} + r - j$ for $j > k + 1$. Taking the union over μ like this, we have*

$$(6.5) \quad \mathcal{C}_{\lambda+\rho}^{(w)} = \bigcup_{\mu} \mathcal{C}_\mu.$$

Proof. Most of the work for the proof has been done in Section 4. Note that by Proposition 11, \mathcal{C}_μ is a component of $\mathcal{C}_{\lambda+\rho}$ in the decomposition (4.20) if and only if μ and $\lambda + \rho$ interleave. Further, it was noted in Remark 4 that the last r segments of the BZL path agree for every element $v \in \mathcal{C}_\mu$. Thus, by Remark 7, either every vertex of \mathcal{C}_μ is contained in $\mathcal{C}_{\lambda+\rho}^{(w)}$, or none of them are. To characterize the weights μ such that $\mathcal{C}_\mu \subseteq \mathcal{C}_{\lambda+\rho}^{(w)}$, recall that by Proposition 11, $v \in \mathcal{C}_{\lambda+\rho}$ belongs to \mathcal{C}_μ if and only if the top two rows of $\mathfrak{T}(v)$ are $\lambda + \rho$ and μ . Further, by Proposition 10, $\Gamma(\mathfrak{T}(v)) = BZL(v)$, so the top row of $BZL(v)$ is given by

$$b_{(r)_2+j}(v) = \lambda_{j+1} + r - j - \mu_j \text{ for every } 1 \leq j \leq r.$$

In particular, we have $b_i(v) = 0$ for every $i > \ell(w) = \binom{r}{2} + k + 1$ if $\lambda_{j+1} + r - j = \mu_j$ holds for every $j > k + 1$. \square

The following proposition relates the right-hand side of Theorem 1 to similar sums over complete highest-weight crystals of a lower rank. It will be key in the proof of that theorem, but is at this point a straightforward consequence of parts (b) and (c) of Proposition 11, and Proposition 15.

Proposition 16. *Using the notation of Proposition 11 and Proposition 15, let $v_*^{(\mu)}$ denote the lowest element of the crystal \mathcal{C}_μ . Write $\rho_{r-1} = (r-1, \dots, 1, 0)$. Then we have*

$$(6.6) \quad \mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n, \lambda+\rho)}(v) \cdot \mathbf{x}^{\text{wt}(v)} = \sum_{\mu} G^{(n, \lambda+\rho)}(v_*) \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} \cdot \left(\mathbf{y}^{-w_0^{(r-1)}(\rho_{r-1})} \cdot \sum_{v \in \mathcal{C}_\mu} G^{(n, \mu)}(v) \cdot \mathbf{y}^{\text{wt}_\mu(v)} \right).$$

Here the sum is over all $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ that interleave with $\lambda + \rho$ and $\mu_j = \lambda_{j+1} + r - j$ for $j > k + 1$.

Since all the notation necessary for our statements is a lot to keep track of, we present an example.

6.1. An example. Take $r = 2$, $\lambda + \rho = (3, 1, 0)$ and $w = \sigma_1 \sigma_2$ (so $k = 0$). Then $\mathcal{C}_{\lambda+\rho}^{(w)} = \mathcal{C}_{(3,1,1)}^{(\sigma_1 \sigma_2)}$ is as in Figure 5. Under the branching (4.20) it is the union of three crystals of type A_1 , of highest weight $(3, 0)$, $(2, 0)$ and $(1, 0)$, respectively.

Determining the highest weights $(3, 0)$, $(2, 0)$ and $(1, 0)$ is easy from Proposition 10 and (6.4). Recall that a vertex belongs to the Demazure crystal $\mathcal{C}_{\lambda+\rho}^{(w)}$ if $BZL(v) = \Gamma(\mathfrak{T}(v))$ has zeros in the last $l(w_0) - l_w = k + 1$ places in the first row. We have $\lambda + \rho = (a_{00}, a_{01}, a_{02}) = (3, 1, 0)$. This means that the component \mathcal{C}_μ of (4.20) belongs to $\mathcal{C}_{(3,1,1)}^{(\sigma_1 \sigma_2)}$ if $\mu = (\mu_1, \mu_2)$ satisfies $\Gamma_2 = \mu_2 - a_{0,2} = 0$, hence $\mu_2 = 0$. Figure 6 shows the three components of $\mathcal{C}_{(3,1,1)}^{(\sigma_1 \sigma_2)}$. Let $n = 1$. The Gelfand-Tsetlin coefficients assigned to the vertices of $\mathcal{C}_{(3,1,1)}^{(\sigma_1 \sigma_2)}$ can be read off of Figure 7.

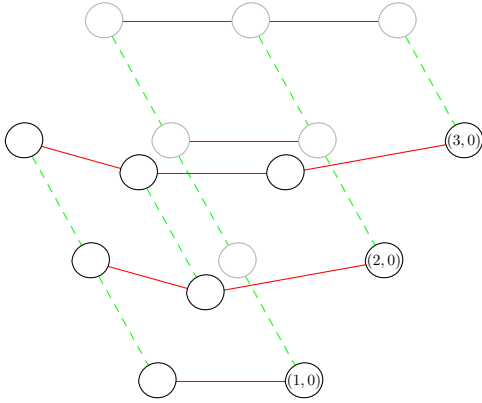


FIGURE 6. Branching

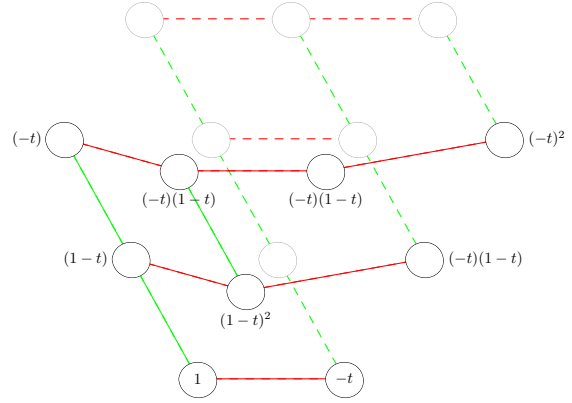


FIGURE 7. Coefficients

Let us restrict our attention to the top A_1 string, $\mathcal{C}_{(3,0)} \subseteq \mathcal{C}_{(3,1,0)}^{(\sigma_1 \sigma_2)}$. For the vertices $v \in \mathcal{C}_{(3,0)} \subseteq \mathcal{C}_{(3,1,0)}^{(\sigma_1 \sigma_2)}$ we have

$$\mathfrak{T}(v) = \begin{pmatrix} 3 & 1 & 0 \\ & 3 & 0 \\ & & a_{22} \end{pmatrix}, \quad \text{and} \quad \Gamma(\mathfrak{T}(v)) = BZL(v) = \begin{bmatrix} 3 & 0 \\ & \Gamma_{22} \end{bmatrix},$$

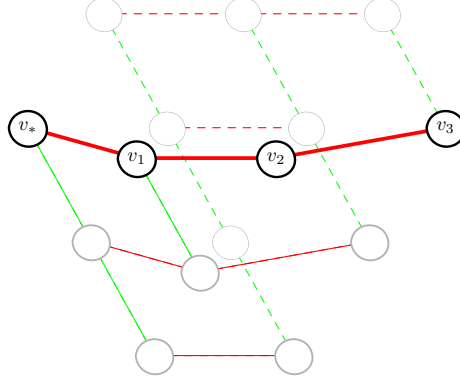
where $\Gamma_{22} = a_{22} - a_{12} = a_{22}$.

The table below show the vertices of this string, and the corresponding Gelfand-Tsetlin coefficients.

v	$\text{wt}(v)$	$G^{(1, \lambda + \rho)}(v)$	$G^{(1, \mu)}(v)$
v_*	$(0, 3, 1)$	$-t$	1
v_1	$(1, 2, 1)$	$(-t)(1-t)$	$1-t$
v_2	$(2, 1, 1)$	$(-t)(1-t)$	$1-t$
v_3	$(3, 0, 1)$	$(-t)(-t)$	$-t$

Figure 8 shows the vertices labeled within $\mathcal{C}_{(3,0)} \subseteq \mathcal{C}_{(3,1,0)}^{(\sigma_1 \sigma_2)} \subseteq \mathcal{C}_{(3,1,0)}$.

We see that if $\mu = (3, 0)$ we have $d(\mu) = 3$ and $d(\lambda + \rho) = 4$. Further, $G^{(n, \lambda + \rho)}(v) = G^{(n, \mu)}(v) \cdot G^{(n, \lambda + \rho)}(v_*)$ holds for $v \in \mathcal{C}_{(3,0)}$.

FIGURE 8. $\mathcal{C}_{(3,0)}$ within $\mathcal{C}_{(3,1,0)}^{(\sigma_1\sigma_2)}$.

6.2. Notation for branching of the Demazure crystal. Proposition 15 and Proposition 16 implies that when dealing with sums over the Demazure crystal $\mathcal{C}_{\lambda+\rho}^{(w)}$, one can treat the components \mathcal{C}_μ as units. Computations for the proof of Theorem 1 will often only involve $\lambda + \rho$ and μ , the top two rows of the Gelfand-Tsetlin patterns parameterizing $\mathcal{C}_\mu \subseteq \mathcal{C}_{\lambda+\rho}^{(w)}$. The following notation and terminology serves to facilitate these computations.

Recall that the first row of the Γ -array, $(\Gamma_{11}, \dots, \Gamma_{1r})$ is the same for every element of a component $\mathcal{C}_\mu \subseteq \mathcal{C}_{\lambda+\rho}^{(w)}$. With λ fixed, we phrase our notation in terms of this r -tuple. Lemma 17 justifies the choices made in the following definition.

Definition 7. Let $\lambda = (\lambda_1, \dots, \lambda_r, \lambda_{r+1})$, $\mu = (\mu_1, \dots, \mu_r)$, and $\Gamma = (\Gamma_{11}, \dots, \Gamma_{1r}) \in \mathbb{Z}^r$. (Set $\Gamma_{1,r+1} := 0$.) We call Γ λ -admissible if

$$(6.7) \quad \Gamma_{1,j+1} \leq \Gamma_{1,j} \leq \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1 \text{ for every } 1 \leq j \leq r.$$

We call Γ (λ, k) -admissible if Γ is λ -admissible and $\Gamma_{i,j} = 0$ for $k+1 < j$. We call Γ *non-strict* if

$$(6.8) \quad \Gamma_{1,j-1} = \Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1 \text{ for at least one } 1 < j \leq r,$$

and *strict* if it is not non-strict.

We define a weight function and Gelfand-Tsetlin coefficients for a Γ r -tuple that is λ -admissible:

$$(6.9) \quad \text{wt}^{(\lambda)}(\Gamma) = (\lambda_{r+1} + \Gamma_{1,r}, \lambda_r + \Gamma_{1,r-1} - \Gamma_{1,r}, \dots, \lambda_2 + \Gamma_{11} - \Gamma_{1,2}, \lambda_1 - \Gamma_{11});$$

$$(6.10) \quad G_1^{(\lambda)}(\Gamma) = G_1^{(n,\lambda)}(\Gamma) = \prod_{j=1}^r g_{1j}^{(\lambda)}(\Gamma) = \prod_{j=1}^r g_{1j}^{(n,\lambda)}(\Gamma),$$

$$g_{1j}^{(n,\lambda)}(\Gamma) = \begin{cases} 1 & \Gamma_{1,j+1} = \Gamma_{1,j} < \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1 \\ h^b(\Gamma_{1,j}) & \Gamma_{1,j+1} < \Gamma_{1,j} < \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1 \\ g^b(\Gamma_{1,j}) & \Gamma_{1,j+1} < \Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1 \\ 0 & \text{otherwise} \end{cases}$$

For convenience, we say Γ is associated to λ and μ and write $\Gamma = \Gamma(\lambda, \mu)$ if

$$(6.11) \quad \Gamma_{1,j} - \Gamma_{1,j+1} = \mu_j - (\lambda_{j+1} + r - j).$$

is satisfied. This is the case if Γ is the first row of an array $\Gamma(\mathfrak{T})$ of a pattern \mathfrak{T} with top two rows $\lambda + \rho_r$ and μ .

Parts (i) – (v) of the following lemma justify the choices in Definition 7. Part (vi) will be convenient in later computations.

Lemma 17. *Let $\lambda = (\lambda_1, \dots, \lambda_r, \lambda_r + 1)$, $\mu = (\mu_1, \dots, \mu_r)$ and $\Gamma = (\Gamma_{11}, \dots, \Gamma_{1r}) = \Gamma(\lambda, \mu)$ associated to λ and μ by (6.11). Then the following statements hold.*

- (i) *The tuple Γ is λ -admissible if and only if the weights $\lambda + \rho_r$ and μ interleave.*
- (ii) *Let w be as in (6.2). Then Γ is (λ, k) -admissible if and only if $\mathcal{C}_\mu \subseteq \mathcal{C}_{\lambda+\rho}^{(w)}$.*
- (iii) *The tuple Γ is strict if and only if μ is strongly dominant.*
- (iv) *Let v_* be the lowest element of a component $\mathcal{C}_\mu \subseteq \mathcal{C}_{\lambda+\rho}^{(w)}$. Then*

$$(6.12) \quad \text{wt}(v_*) - w_0^{(r)}(\rho_r) = \text{wt}^{(\lambda)}(\Gamma).$$

- (v) *Let v_* be the lowest element of a component $\mathcal{C}_\mu \subseteq \mathcal{C}_{\lambda+\rho}^{(w)}$. Then*

$$(6.13) \quad G^{(n, \lambda+\rho)}(v_*) = \begin{cases} G_1^{(n, \lambda)}(\Gamma) & \text{if } \mu \text{ is strongly dominant;} \\ 0 & \text{otherwise.} \end{cases}$$

- (vi) *With the notation as above, we have*

$$(6.14) \quad \mathbf{x}^{\text{wt}^{(\lambda)}(\Gamma)} = \mathbf{y}^{w_0^{(r-1)}(\mu - \rho_{r-1})} \cdot x_{r+1}^{d(\lambda+\rho) - d(\mu) - r}.$$

Proof. Note first that the condition (6.11) is satisfied exactly if $\Gamma = (\Gamma_{11}, \dots, \Gamma_{1r})$ is the first row of the array $\Gamma(\mathfrak{T})$, When \mathfrak{T} is a pattern with top rows $\lambda + \rho$ and μ . With this observation, the proof is straightforward from Propositions 12, 15 and 16. For (i), we have that by (6.11),

$$\lambda_j + r - j + 1 \geq \mu_j \geq \lambda_{j+1} + r - j \iff \lambda_j - \lambda_{j+1} + 1 \geq \Gamma_{1,j} - \Gamma_{1,j+1} \geq 0.$$

Again by (6.11) we have that Γ is (λ, k) -admissible if and only if for any $k + 1 < j$ we have

$$\Gamma_{1,j} = \mu_j - (\lambda_{j+1} + r - j) + \Gamma_{1,j+1} = \Gamma_{1,j+1} = 0 \iff \mu_j = \lambda_{j+1} + r - j.$$

By Proposition 15, this is equivalent to $\mathcal{C}_\mu \subseteq \mathcal{C}_{\lambda+\rho}^{(w)}$. This proves (ii). Part (iii) is true because

$$\mu_{j-1} = \mu_j \iff \mu_{j-1} = \lambda_j + r - j + 1 = \mu_j$$

and by (6.11)

$$\Gamma_{1,j-1} = \Gamma_{1,j} \iff \Gamma_{1,j-1} - \Gamma_{1,j} = \mu_{j-1} - (\lambda_j + r - j + 1) = 0;$$

$$\Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1 \iff \Gamma_{1,j} - \Gamma_{1,j+1} = \mu_j - (\lambda_{j+1} + r - j) = \lambda_j - \lambda_{j+1} + 1.$$

For part (iv), recall that by part (c) of Proposition 10, if v_{lowest} is the lowest element of $\mathcal{C}_{\lambda+\rho}$, then $\text{wt}(v) - \text{wt}(v_{\text{lowest}})$ can be expressed from the entries $b_i(v)$ of $BZL(v)$. We have $\text{wt}(v_{\text{lowest}}) = w_0^{(r)}(\lambda + \rho)$. Furthermore, by Remark 4, for the lowest element v_* of a component \mathcal{C}_μ , $b_i(v_*) = 0$ if $i \leq \binom{r}{2}$. Further, since Γ is the top row of $BZL(v_*) = \Gamma(\mathfrak{T}(v_*))$, we also have $b_i(v_*) = \Gamma_{1,i - \binom{r}{2}}$. Now (4.14) implies that

$$\text{wt}(v_*) - w_0^{(r)}(\rho) = w_0^{(r)}(\lambda) + \sum_{j=1}^r \Gamma_{1,j} \cdot \alpha_{r+1-j} = \text{wt}^{(\lambda)}(\Gamma).$$

To prove (v), recall that by Remark 3 if μ is not strongly dominant (i.e. Γ is nonstrict), then $G^{(n, \lambda + \rho)}(v_*) = 0$. Furthermore, if μ is strongly dominant, then by (4.25), the Gelfand-Tsetlin coefficient corresponding to v_* only depends on the first row of the BZL -array:

$$G^{(n, \lambda + \rho)}(\mathfrak{T}(v_*)) = \prod_{1 \leq j \leq r} g_{1j}^{n, \lambda + \rho}(\mathfrak{T}(v_*)).$$

Now since the first two rows of $\mathfrak{T}(v_*)$ are $\lambda + \rho$ and μ , and Γ is the first row of $\Gamma(\mathfrak{T}(v_*))$, the statement follows immediately from comparing (6.10) and (4.16).

Finally, recall that we write $\mathbf{x} = (x_1, \dots, x_r, x_{r+1})$, $\mathbf{y} = (x_1, \dots, x_r)$ and $d(\lambda)$ (or $d(\mu)$) for the sum of the components of the weight λ (respectively, μ). Note that $\text{wt}_\mu(v_*) = w_0^{(r-1)}(\mu)$ and $d(\lambda + \rho) - d(\mu) = \lambda_1 + r - \Gamma_{11}$. Then from part (iv) above, and part (a) of Proposition 11 we have

$$\begin{aligned} \mathbf{x}^{\text{wt}(\lambda)(\Gamma)} &= \mathbf{x}^{\text{wt}(v_*)} \cdot \mathbf{x}^{-w_0^{(r)}(\rho_r)} \\ &= \mathbf{y}^{\text{wt}_\mu(v_*)} \cdot x_{r+1}^{d(\lambda + \rho) - d(\mu)} \cdot \mathbf{y}^{w_0^{(r-1)}(\rho_{r-1})} \cdot x_{r+1}^{-r} \\ &= \mathbf{y}^{w_0^{(r-1)}(\mu - \rho_{r-1})} \cdot x_{r+1}^{d(\lambda + \rho) - d(\mu) - r}. \end{aligned}$$

□

7. REDUCTION AND PROOF OF THE MAIN THEOREM

We are ready to begin the proof of Theorem 1. The expression on the left-hand side involves a large sum of Demazure-Lusztig operators,

$$(7.1) \quad \left(\sum_{u \leq w_0^{(r)}} \mathcal{T}_u \right) = \Delta_t^{(r)} \cdot \mathcal{D}_{w_0^{(r)}}.$$

The idea behind the proof is that one may replace this expression by progressively simpler ones, eventually reducing the statement of Theorem 1 to the description of the polynomial

$$(7.2) \quad (\mathcal{T}_r \mathcal{T}_{r-1} \cdots \mathcal{T}_1)(\mathbf{x}^{w_0(\lambda)}).$$

The statement describing the polynomial (7.2) is then proved by induction in Section 8.

In the proof we restrict our attention to the case where the Weyl group element w (a beginning section of $w_0^{(r)}$) has length at least $\binom{r}{2} + 1$. This leads to no loss of generality by the following remark.

Remark 8. The statement of Theorem 1 in type A_r but for $\ell(w) \leq \binom{r}{2}$ is equivalent to an instance of the theorem in type A_{r-1} . Let λ be as in the statement of the theorem, $\lambda' = (\lambda_2, \lambda_3, \dots, \lambda_{r+1})$ and suppose $\ell(w) \leq \binom{r}{2}$. Then in fact w is a beginning section of $w_0^{(r-1)}$. The statement (2.4) for type A_r , λ and w is the analogous statement for type A_{r-1} , λ' and w , except both sides are multiplied by $x_{r+1}^{\lambda_1}$. On the left-hand side, $\mathcal{T}_1, \dots, \mathcal{T}_{r-1}$ all commute with multiplication by x_{r+1} . As for the right-hand side, in the decomposition (4.20), $\mathcal{C}_{\lambda + \rho}^{(w)}$ is contained in the component $\mathcal{C}_{\lambda' + \rho_{r-1}}$ of the lowest element, $v_{\text{lowest}} \in \mathcal{C}_{\lambda + \rho}$. We have $G^{\lambda + \rho}(v_{\text{lowest}}) = 1$. The statement now follows from Proposition 11.

Hence from now on, we shall assume that w is as in (6.2). Recall that w is fixed by a choice of the pair r, k where $0 \leq k < r$. Call the statement of Theorem 1 for such a fixed w and fixed n (but for any dominant, effective weight λ) $IW_{r,k}^{(n)}$. Proving $IW_{r,k}^{(n)}$ for any pair $0 \leq k < r$ proves Theorem 1.

Theorem 2 is the special case of Theorem 1 where $w = w_0^{(r)}$, i.e. $k = r - 1$. We will sometimes use the notation $Tok_r^{(n)} = IW_{r,r-1}^{(n)}$.

Remark 9. Much, but not all of the notation introduced above, in previous chapters, and in what follows, depends on the value of n . In particular the meaning of \mathcal{D}_i , \mathcal{T}_i , (the action of) σ_i , $G^{(\lambda,w)}(v)$, $Tok_r^{(n)}$ and $IW_{r,k}^{(n)}$ depend on n , but w_0 , W , $\mathcal{C}_{\lambda+\rho}^{(w)}$ and $\text{wt}(v)$ do not. We will usually suppress n from the notation. When reading the statements and proofs below, one should keep in mind that the meaning varies with n . The entire argument of the proof is about a(n arbitrarily) fixed n .

The reduction of $IW_{r,k}$ to the simpler statement is itself an induction by r . We will phrase two more statements, $M_{r,k}$ (Proposition 20) and $N_{r,k}$ (Proposition 21). These involve smaller expressions of Demazure-Lusztig operators on the left hand side, and make use of the notation of Definition 7. $N_{r,r-1}$ describes the polynomial in (7.2).

The technical ingredients of the reduction are stated as lemmas or auxiliary propositions along the way. The proof, using these auxiliary statements, is in section 7.2. The last ingredient is the proof of Proposition 24, which is a rather technical induction, and forms the contents of Section 8.

7.1. Auxiliary statements. First we rewrite both sides of $IW_{r,k}$ in terms of the operator appearing in Tok_{r-1} , of the form (7.1) For the left-hand side, this is accomplished by the following lemma. It is really just a statement about the Bruhat-order; we omit the proof.

Lemma 18. *Let $w = w_0^{(r-1)} \cdot \sigma_r \cdots \sigma_{r-k}$. Then*

$$(7.3) \quad \sum_{u \leq w} \mathcal{T}_u = \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \cdot (1 + \mathcal{T}_r + \mathcal{T}_r \mathcal{T}_{r-1} + \cdots + \mathcal{T}_r \cdots \mathcal{T}_{r-k}).$$

The first factor on the right-hand side of (7.3) is

$$(7.4) \quad \sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u,$$

the operator on the left-hand side of Tok_{r-1} . By Theorem 6, it is equal to $\Delta_t^{(r-1)} \cdot \mathcal{D}_{w_0^{(r-1)}}$.

The second factor will appear as the operator in $M_{r,k}$ (Proposition 20).

The following Proposition reproduces the right-hand side of $IW_{r,k}$ in terms of the operator (7.4). It is a consequence of Proposition 16 and Lemma 17. It is proved in section 7.3.

Proposition 19. *Assume $IW_{r-1,r-2}(=Tok_{r-1})$ holds. Then we have*

$$(7.5) \quad \mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\text{wt}(v)} = \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \sum_{\substack{\Gamma=(\Gamma_{11},\dots,\Gamma_{1r}) \\ \Gamma \text{ } (\lambda,k)\text{-admissible}}} G_1^\lambda(\Gamma) \cdot \mathbf{x}^{\text{wt}(\lambda)(\Gamma)}$$

Lemma 18 and Proposition 19 together produce both sides of $IW_{r,k}$ as the operator in (7.4) applied to a polynomial. The fact that the “inputs” are the same up to annihilation by this operator is the statement that we will call $M_{r,k}$. The next proposition phrases the statement $M_{r,k}$ explicitly for any $0 \leq k < r$.

Proposition 20. *Let $\lambda = (\lambda_1, \dots, \lambda_r, \lambda_{r+1})$ be any dominant weight, $0 \leq k < r$ integers. Then we have*

$$(7.6) \quad (1 + \mathcal{T}_r + \mathcal{T}_r \mathcal{T}_{r-1} + \dots + \mathcal{T}_r \dots \mathcal{T}_{r-k}) \mathbf{x}^{w_0(\lambda)} \equiv \sum_{\substack{\Gamma = (\Gamma_{11}, \dots, \Gamma_{1r}) \\ \Gamma \text{ } (\lambda, k)\text{-admissible}}} G_1^\lambda(\Gamma) \cdot \mathbf{x}^{\text{wt}^{(\lambda)}(\Gamma)}$$

Here \equiv means that the difference of the left and right hand side is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$. Call this statement (that (7.6) holds for any λ dominant weight) $M_{r,k}$.

The statement $M_{r,k}$ lends itself to an obvious simplification. On the left hand side, there is a sum of $k+1$ strings of Demazure-Lusztig operators. The statement $N_{r,k}$ involves only one of them.

Proposition 21. *Let $\lambda = (\lambda_1, \dots, \lambda_r, \lambda_{r+1})$ be any dominant weight, $0 \leq k < r$ integers. Then we have*

$$(7.7) \quad (\mathcal{T}_r \dots \mathcal{T}_{r-k}) \mathbf{x}^{w_0(\lambda)} \equiv \sum_{\substack{\Gamma = (\Gamma_{11}, \dots, \Gamma_{1r}) \\ \Gamma \text{ } (\lambda, k)\text{-admissible} \\ \Gamma_{1,k+1} \neq 0}} G_1^\lambda(\Gamma) \cdot \mathbf{x}^{\text{wt}^{(\lambda)}(\Gamma)}$$

Here \equiv means that the difference of the left and right hand side is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$. Call this statement (that (7.7) holds for any λ dominant weight) $N_{r,k}$.

Remark 10. Note that in both $M_{r,k}$ and $N_{r,k}$, λ is not required to be effective, i.e. it may have negative components. We may however always assume that it is effective, replacing λ by $\kappa = (\lambda_1 + K, \dots, \lambda_r + K, \lambda_{r+1} + K)$. This can be done because as an operator, multiplication by $(x_1 \cdot x_2 \dots x_{r+1})^K$ commutes with \mathcal{T}_i and \mathcal{D}_i for any $1 \leq i \leq r$, and

$$\mathbf{x}^{w_0(\kappa)} = \mathbf{x}^{w_0(\lambda)} \cdot (x_1 \cdot x_2 \dots x_{r+1})^K, \quad \mathbf{x}^{\text{wt}^\kappa(\Gamma)} = \mathbf{x}^{\text{wt}^\lambda(\Gamma)} \cdot (x_1 \cdot x_2 \dots x_{r+1})^K.$$

The following lemma is straightforward.

Lemma 22. *Proposition 21 implies Proposition 20. That is, we have*

$$(7.8) \quad \forall r, k \quad N_{r,k} \implies \forall r, k \quad M_{r,k}.$$

As a last step in the sequence of replacing Theorem 1 with simpler statements, we note that in the statement $N_{r,k}$, the parameter k is the interesting one. This is the content of Lemma 23 below. The proof is straightforward by renaming variables, and keeping in mind that multiplication by x_i commutes with \mathcal{T}_j and \mathcal{D}_j if $i \notin \{j, j+1\}$.

Lemma 23. *If $N_{k+1,k}$ is true, then $N_{r,k}$ is true for every $r > k$. In fact, $N_{k+1,k}$ implies a slightly stronger statement than $N_{r,k}$: the difference of the left-hand side and the right-hand side is annihilated not just by $\mathcal{D}_{w_0^{(r-1)}}$, but by the Demazure-operator corresponding to the long word in the group $\langle \sigma_{r-k}, \sigma_{r-k+1}, \dots, \sigma_{r-1} \rangle$.*

The statement $N_{k+1,k}$ will be proved in Section 8 as Proposition 24. We are now ready to give the proof of Theorem 1.

7.2. The proof of Theorem 1. By Proposition 24 (proved in Section 8), we have that $N_{k+1,k}$ holds for any nonnegative k . By Lemma 23, this implies that $N_{r,k}$ holds for any pair of integers $0 \leq k < r$, i.e. Proposition 21 is true. By Lemma 22, this proves Proposition 20, i. e. $M_{r,k}$ for any pair of integers $0 \leq k < r$.

We prove $IW_{r,k}$ for any pair of integers $0 \leq k < r$ by induction on r .

To start, notice that both $M_{1,0}$ and $IW_{1,0}$ state that if $\lambda_1 \geq \lambda_2$, then

$$(7.9) \quad (1 + T_1)x_1^{\lambda_2}x_2^{\lambda_1} = \sum_{\Gamma_{11}=0}^{\lambda_1-\lambda_2+1} G_1^{(\lambda)}(\Gamma_{11}) \cdot x_1^{\lambda_2+\Gamma_{11}}x_2^{\lambda_2-\Gamma_{11}} \\ = \frac{1}{x_2} \sum_{\mathfrak{T}} G(\mathfrak{T}) \cdot \mathbf{x}^{wt(\mathfrak{T})},$$

where the sum is over all Gelfand-Tsetlin patterns \mathfrak{T} of the form

$$\mathfrak{T} = \begin{pmatrix} \lambda_1 + 1 & & \lambda_2 \\ & \lambda_2 + \Gamma_{11} & \end{pmatrix}.$$

Thus $IW_{1,0}$ is the same as $M_{1,0}$, and in particular, $IW_{r,k}$ is true if $r = 1$.

Now let $r > 1$, $0 \leq k < r$ and assume that $IW_{r-1,r-2} = Tok_{r-1}$ is true. We know $M_{r,k}$ holds, hence

$$(7.10) \quad (1 + \mathcal{T}_r + \mathcal{T}_r\mathcal{T}_{r-1} + \cdots + \mathcal{T}_r \cdots \mathcal{T}_{r-k})\mathbf{x}^{w_0(\lambda)} \equiv \sum_{\substack{\Gamma=(\Gamma_{11},\dots,\Gamma_{1r}) \\ \Gamma(\lambda,k)\text{-admissible}}} G_1^\lambda(\Gamma) \cdot \mathbf{x}^{wt^{(\lambda)}(\Gamma)},$$

i.e. the difference of the two sides of (7.10) is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$. By Theorem 6, the difference is then also annihilated by

$$\Delta_t^{(r-1)} \cdot \mathcal{D}_{w_0^{(r-1)}} = \sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u.$$

That is, we have

$$(7.11) \quad \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) (1 + \mathcal{T}_r + \mathcal{T}_r\mathcal{T}_{r-1} + \cdots + \mathcal{T}_r \cdots \mathcal{T}_{r-k})\mathbf{x}^{w_0(\lambda)} \\ = \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \sum_{\substack{\Gamma=(\Gamma_{11},\dots,\Gamma_{1r}) \\ \Gamma(\lambda,k)\text{-admissible}}} G_1^\lambda(\Gamma) \cdot \mathbf{x}^{wt^{(\lambda)}(\Gamma)}.$$

Rewriting the left hand side of (7.11) by Lemma 18, and the right hand side by Proposition 19 we arrive at

$$(7.12) \quad \left(\sum_{u \leq w_0^{(r)}} \mathcal{T}_u \right) \mathbf{x}^{w_0(\lambda)} = \mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{wt(v)}.$$

This is exactly the statement $IW_{r,k}$.

Thus $IW_{r,k}$ is true for any pair of integers $0 \leq k < r$. By Remark 8, this completes the proof of Theorem 1. \square

7.3. The proof of Proposition 19. We prove that if Tok_{r-1} (equivalently, $IW_{r-1,r-2}$) holds, then

$$(7.13) \quad \mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\text{wt}(v)} = \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \sum_{\substack{\Gamma = (\Gamma_{11}, \dots, \Gamma_{1r}) \\ \Gamma \text{ } (\lambda, k)\text{-admissible}}} G_1^\lambda(\Gamma) \cdot \mathbf{x}^{\text{wt}(\lambda)(\Gamma)}.$$

By Proposition 16, we have

$$(7.14) \quad \mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n,\lambda+\rho)}(v) \cdot \mathbf{x}^{\text{wt}(v)} = \sum_{\mu} G^{(n,\lambda+\rho)}(v_*) \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} \cdot \left(\mathbf{y}^{-w_0^{(r-1)}(\rho_{r-1})} \cdot \sum_{v \in \mathcal{C}_{\mu}} G^{(n,\mu)}(v) \cdot \mathbf{y}^{\text{wt}_{\mu}(v)} \right).$$

Here the sum is over all $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ that interleave with $\lambda + \rho$ and $\mu_j = \lambda_{j+1} + r - j$ for $j > k + 1$. We claim that

$$(7.15) \quad \mathbf{y}^{-w_0^{(r-1)}(\rho_{r-1})} \cdot \sum_{v \in \mathcal{C}_{\mu}} G^{(n,\mu)}(v) \cdot \mathbf{y}^{\text{wt}_{\mu}(v)} = \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \mathbf{y}^{w_0^{(r-1)}(\mu - \rho_{r-1})}.$$

Since μ interleaves with $\lambda + \rho$, it is dominant and effective. We distinguish between two cases according to whether μ is strongly dominant or not.

If μ is strongly dominant, then $\mu - \rho_{r-1}$ is dominant and effective. In this case (7.15) is the statement Tok_{r-1} ($IW_{r-1,r-2}$) for the weight $\mu - \rho_{r-1}$, hence it is true by the assumption that Tok_{r-1} holds.

Suppose now that μ is not strongly dominant, i.e. we have $\mu_j = \mu_{j+1}$ for some $1 \leq j \leq r-1$. We show that then both sides of (7.15) are zero. The left hand side is zero by Remark 3. We show that the operator on the right hand side of (7.15) annihilates the monomial $\mathbf{y}^{w_0^{(r-1)}(\mu - \rho_{r-1})}$. By Theorem 6, this operator is $\Delta_t^{(r-1)} \cdot \mathcal{D}_{w_0^{(r-1)}}$. Since $w_0^{(r-1)}$ is the long element in the Weyl group generated by $\sigma_1, \dots, \sigma_{r-1}$, by Lemma 8 it suffices to prove $\mathcal{D}_i \mathbf{y}^{w_0^{(r-1)}(\mu - \rho_{r-1})} = 0$ for at least one index $1 \leq i = r - j \leq r - 1$. Let $i = r + 1 - (j + 1) = r - j$, $i + 1 = r - j + 1$. Then in the monomial $\mathbf{y}^{w_0^{(r-1)}(\mu - \rho_{r-1})}$, x_i appears with exponent $\mu_{j+1} - r + j + 1 = \mu_j - r + j + 1$, while x_{i+1} appears with exponent $\mu_j - r + j$. Thus, by Corollary 9, indeed $\mathcal{D}_i \mathbf{y}^{w_0^{(r-1)}(\mu - \rho_{r-1})} = 0$. Thus the right hand side of (7.15) is indeed zero if μ is not strongly dominant.

Having established (7.15), we have that

$$\begin{aligned}
 (7.16) \quad \mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n, \lambda+\rho)}(v) \cdot \mathbf{x}^{\text{wt}(v)} &= \sum_{\mu} G^{(n, \lambda+\rho)}(v_*) \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} \\
 &\cdot \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})} \\
 &= \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \\
 &\sum_{\mu} G^{(n, \lambda+\rho)}(v_*) \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} \cdot \mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})}.
 \end{aligned}$$

Here the second equation is true because multiplication by x_{r+1} commutes with \mathcal{T}_u for every $u \leq w_0^{(r-1)}$. By Lemma 17, we have that if $\Gamma = \Gamma(\lambda, \mu) = (\Gamma_1, \dots, \Gamma_r)$ as in Definition 7, then Γ is (λ, k) admissible exactly if μ appears in the summation in (7.14), furthermore

$$G^{(n, \lambda+\rho)}(v_*) = \begin{cases} G_1^{(n, \lambda)}(\Gamma) & \text{if } \mu \text{ is strongly dominant;} \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathbf{y}^{w_0^{(r-1)}(\mu-\rho_{r-1})} \cdot x_{r+1}^{d(\lambda+\rho)-d(\mu)-r} = \mathbf{x}^{\text{wt}(\lambda)(\Gamma)}.$$

Thus we may rewrite (7.16) further as

$$(7.17) \quad \mathbf{x}^{-w_0(\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}^{(w)}} G^{(n, \lambda+\rho)}(v) \cdot \mathbf{x}^{\text{wt}(v)} = \left(\sum_{u \leq w_0^{(r-1)}} \mathcal{T}_u \right) \sum_{\substack{\Gamma = (\Gamma_{11}, \dots, \Gamma_{1r}) \\ \Gamma \text{ } (\lambda, k)\text{-admissible}}} G_1^{(n, \lambda)}(\Gamma) \cdot \mathbf{x}^{\text{wt}(\lambda)(\Gamma)}.$$

This is exactly (7.13); the proof of Proposition 19 is complete.

8. PROOF OF THE STATEMENT $N_{r, r-1}$

In Section 7, the proof of Theorem 1 and Theorem 2 was reduced to describing the action of the string of Demazure-Lusztig operators $\mathcal{T}_r \dots \mathcal{T}_1$ on a monomial, i.e. the statement $N_{r, r-1}$. This section consists of the proof of the statement $N_{r, r-1}$. We recall the statement in Proposition 24 below.

Proposition 24. *Let $\lambda = (\lambda_1, \dots, \lambda_r, \lambda_{r+1})$ be any dominant weight. Then we have*

$$(8.1) \quad (\mathcal{T}_r \dots \mathcal{T}_1) \mathbf{x}^{w_0(\lambda)} \equiv \sum_{\substack{\Gamma = (\Gamma_{11}, \dots, \Gamma_{1r}) \\ \Gamma \text{ } \lambda\text{-admissible} \\ \Gamma_{1, r} \neq 0}} G_1^\lambda(\Gamma) \cdot \mathbf{x}^{\text{wt}(\lambda)(\Gamma)}.$$

Here \equiv means that the difference of the left and right hand side is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$.

Recall that the relevant notation has been introduced in section 6.2. In this section, we use v for denoting $v = t^n = q^{-1}$.

Let us abbreviate both sides of the equation (8.1):

$$(8.2) \quad \mathfrak{L}_r^{(\lambda)}(\mathbf{x}) := (\mathcal{T}_r \cdots \mathcal{T}_1) \mathbf{x}^{w_0(\lambda)} \quad \text{and} \quad \mathfrak{R}_r^{(\lambda)}(\mathbf{x}) := \sum_{\substack{\Gamma = (\Gamma_{11}, \dots, \Gamma_{1r}) \\ \Gamma \text{ } \lambda\text{-admissible} \\ \Gamma_{1,r} \neq 0}} G_1^\lambda(\Gamma) \cdot \mathbf{x}^{\text{wt}^{(\lambda)}(\Gamma)}.$$

The proof is by induction on r . The base case is fairly straightforward using the definitions and Claim 5. We omit this rank one computation and contend that both sides turn out to be equal to the following expression:

$$\mathfrak{L}_1^{(\lambda)}(\mathbf{x}) = \mathfrak{R}_1^{(\lambda)}(\mathbf{x}) = \frac{(1-v) \cdot \mathbf{x}^{n\alpha}}{1 - \mathbf{x}^{n\alpha}} \left(x_1^{\lambda_2} x_2^{\lambda_1} - \mathbf{x}^{-r_n(\lambda_1 - \lambda_2)\alpha} \cdot x_1^{\lambda_1} x_2^{\lambda_2} \right) + v \cdot g_{1+\lambda_1-\lambda_2} \cdot x_1^{\lambda_1+1} x_2^{\lambda_2-1}.$$

For the induction step, we assume that $N_{k+1,k}$ holds for $k < r-1$. The goal is to prove

$$(8.3) \quad \mathfrak{L}_r^{(\lambda)}(\mathbf{x}) \equiv \mathfrak{R}_r^{(\lambda)}(\mathbf{x}), \text{ i.e. } \mathcal{D}_{w_0^{(r-1)}} \left(\mathfrak{L}_r^{(\lambda)}(\mathbf{x}) - \mathfrak{R}_r^{(\lambda)}(\mathbf{x}) \right) = 0.$$

Claim 25. *It suffices to show that if $N_{k+1,k}$ holds for $k < r-1$, then*

$$(8.4) \quad \mathfrak{R}_r^{(\lambda)}(x) \equiv \mathcal{T}_r \left(x_{r+1}^{\lambda_1} \cdot \mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y}) \right), \text{ i.e. } \mathcal{D}_{w_0^{(r-1)}} \left(\mathcal{T}_r \left(x_{r+1}^{\lambda_1} \cdot \mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y}) \right) - \mathfrak{R}_r^{(\lambda)}(\mathbf{x}) \right) = 0.$$

Proof. The proof is straightforward using the fact that multiplication by $x_{r+1}^{\lambda_1}$ commutes with the operators T_1, \dots, T_{r-1} , multiplication by $x_{r+1}^{\lambda_1}$ and \mathcal{T}_r both commute with $\mathcal{D}_{w_0^{(r-2)}}$, and Lemma 8. \square

The remainder of this section will consist of proving (8.4) from the assumption that $N_{k+1,k}$ holds for $k < r-1$. The argument will proceed as follows. After introducing some convenient notation (in 8.1), we shall simplify the induction step (in 8.2). Computing $\mathcal{T}_r \left(x_{r+1}^{\lambda_1} \cdot \mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y}) \right)$ directly, and comparing the result with $\mathfrak{R}_r^{(\lambda)}(\mathbf{x})$, we find that there is a polynomial “left over”. The fact that this polynomial is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$ is called \mathbf{F}_r (Proposition 26). In fact, the computation shows that assuming $N_{r-1,r-2}$, the statements \mathbf{F}_r and $N_{r,r-1}$ are equivalent (Lemma 27). Thus it remains to prove Proposition 26 by showing the statement \mathbf{F}_r ; this is done in section 8.3. This will also (partially) be a proof by induction: by Lemma 27, the assumption of $N_{k+1,k}$ for $k < r-1$ implies in particular that \mathbf{F}_j holds for $j < r$.

8.1. Notation and conventions. The following conventions allow us to relate the statements $N_{r,r-1}$, $N_{r-1,r-2}$ and $N_{r-2,r-3}$ with more transparency. Let weights be denoted by $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{r+1})$, $\mu = (\lambda_2, \dots, \lambda_{r+1})$, and $\nu = (\lambda_3, \dots, \lambda_{r+1})$; the variables with $\mathbf{x} = (x_1, \dots, x_{r-1}, x_r, x_{r+1})$, $\mathbf{y} = (x_1, \dots, x_r)$ and $\mathbf{z} = (x_1, \dots, x_{r-1})$. Furthermore, let $\Gamma' = (\Gamma_{11}, \Gamma_{12}, \Gamma_{13}, \dots, \Gamma_{1r})$, $\Gamma = (\Gamma_{12}, \Gamma_{13}, \dots, \Gamma_{1r})$ and $\Gamma_0 = (\Gamma_{13}, \dots, \Gamma_{1r})$. With this notation, we have that

$$(8.5) \quad \Gamma' \text{ is } \lambda\text{-admissible if and only if } \begin{cases} \Gamma \text{ is } \mu\text{-admissible and} \\ \Gamma_{12} \leq \Gamma_{11} \leq \Gamma_{12} + \lambda_1 - \lambda_2 + 1; \end{cases}$$

and

$$(8.6) \quad G^{(\lambda)}(\Gamma') = g_{11}^{(\lambda)}(\Gamma') \cdot G_1^{(\mu)}(\Gamma), \text{ and } \mathbf{x}^{\text{wt}^{(\lambda)}(\Gamma')} = \mathbf{y}^{\text{wt}^{(\mu)}(\Gamma)} \cdot x_r^{\Gamma_{11}} \cdot x_{r+1}^{\lambda_1 - \Gamma_{11}}.$$

Similarly,

$$(8.7) \quad \Gamma \text{ is } \mu - \text{admissible if and only if } \begin{cases} \Gamma_0 \text{ is } \nu - \text{admissible and} \\ \Gamma_{13} \leq \Gamma_{12} \leq \Gamma_{13} + \lambda_2 - \lambda_3 + 1; \end{cases}$$

and

$$(8.8) \quad G^{(\mu)}(\Gamma) = g_{12}^{(\mu)}(\Gamma) \cdot G_1^{(\nu)}(\Gamma_0), \text{ and } \mathbf{y}^{\text{wt}(\mu)}(\Gamma) = \mathbf{z}^{\text{wt}(\nu)}(\Gamma_0) \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2 - \Gamma_{12}}.$$

Notice that in indexing the μ -admissible vector Γ , we write $g_{12}^{(\mu)}(\Gamma)$ for the Gelfand-Tsetlin coefficient corresponding to the first entry, Γ_{12} . Equations (8.6) and (8.8) are a direct consequence of the notation introduced. In particular, the relationship between the monomials is true even if Γ' (or Γ) is not λ -admissible (respectively, μ -admissible).

We will make repeated use of the following function on pairs of (positive) integers:

$$(8.9) \quad \delta(A, B) = \begin{cases} h^b(A) & \text{if } A < B \\ h^b(A) - 1 & \text{if } A = B \\ 0 & \text{if } A > B \end{cases}$$

We are now ready to tackle the induction step.

8.2. Simplifying the induction step. We set out to prove (8.4). To this end, we first rewrite $\mathcal{T}_r(x_{r+1}^{\lambda_1} \cdot \mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y}))$. By the conventions (8.8), and the fact that \mathcal{T}_r commutes with multiplication by x_1, \dots, x_{r-1} , we have

$$(8.10) \quad \mathcal{T}_r(x_{r+1}^{\lambda_1} \cdot \mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y})) = \sum_{\substack{\Gamma=(\Gamma_{12}, \dots, \Gamma_{1,r}) \\ \Gamma \text{ } \mu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} G_1^{(\mu)}(\Gamma) \cdot \mathbf{z}^{\text{wt}(\nu)}(\Gamma_0) \cdot x_{r-1}^{\Gamma_{12}} \cdot \mathcal{T}_r(x_r^{\lambda_2 - \Gamma_{12}} x_{r+1}^{\lambda_1}).$$

Since $N_{1,0}$ is the base case of the induction, $N_{r,0}$ is true by Lemma 23. Thus we have

$$(8.11) \quad \mathcal{T}_r(x_r^{\lambda_2 - \Gamma_{12}} x_{r+1}^{\lambda_1}) = \sum_{\Gamma_{11}=1}^{\lambda_1 - (\lambda_2 - \Gamma_{12}) + 1} g_{11}^{(\lambda_1, \lambda_2 - \Gamma_{12})}(\Gamma_{11}) x_r^{\lambda_2 + \Gamma_{11} - \Gamma_{12}} x_{r+1}^{\lambda_1 - \Gamma_{11}}.$$

It follows from (6.10) and (8.9) that

$$(8.12) \quad g_{11}^{(\lambda_1, \lambda_2 - \Gamma_{12})}(\Gamma_{11}) - g_{11}^{(\lambda)}(\Gamma') = \delta(\Gamma_{11}, \Gamma_{12}).$$

Now we may substitute (8.11) and (8.12) into (8.10), and use the conventions (8.5), (8.6) to conclude that

$$\mathcal{T}_r(x_{r+1}^{\lambda_1} \cdot \mathfrak{R}_{r-1}^{(\mu)}(\mathbf{y})) = \mathfrak{R}_r^{(\lambda)}(x) + \sum_{1 \leq \Gamma_{11}} x_{r+1}^{\lambda_1 - \Gamma_{11}} \cdot \sum_{\substack{\Gamma=(\Gamma_{12}, \dots, \Gamma_{1,r}) \\ \Gamma \text{ } \mu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} \delta(\Gamma_{11}, \Gamma_{12}) G_1^{(\mu)}(\Gamma) \mathbf{y}^{\text{wt}(\mu)}(\Gamma) x_r^{\Gamma_{11}}.$$

Thus proving (8.4) is equivalent to showing that

$$(8.13) \quad \sum_{1 \leq \Gamma_{11}} x_{r+1}^{\lambda_1 - \Gamma_{11}} \cdot \sum_{\substack{\Gamma=(\Gamma_{12}, \dots, \Gamma_{1,r}) \\ \Gamma \text{ } \mu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} \delta(\Gamma_{11}, \Gamma_{12}) \cdot G_1^{(\mu)}(\Gamma) \cdot \mathbf{y}^{\text{wt}(\mu)}(\Gamma) \cdot x_r^{\Gamma_{11}} \equiv 0,$$

i.e. it is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$. Since multiplication by x_{r+1} commutes with $\mathcal{D}_{w_0^{(r-1)}}$, (8.13) is equivalent to showing that the part corresponding to a fixed power of x_{r+1} is annihilated

by $\mathcal{D}_{w_0^{(r-1)}}$. This motivates the following notation. Let a be a positive integer, and μ, \mathbf{y} as in Section 8.1.

$$(8.14) \quad F_{\mu,a}(\mathbf{y}) = \sum_{\substack{\Gamma=(\Gamma_{12},\dots,\Gamma_{1r}) \\ \Gamma \text{ } \mu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} \delta(a, \Gamma_{12}) \cdot G_1^{(\mu)}(\Gamma) \cdot \mathbf{y}^{\text{wt}(\mu)(\Gamma)} \cdot x_r^a.$$

Proposition 26. *Let $\mu = (\lambda_2, \lambda_3, \dots, \lambda_{r+1})$ be any dominant weight. Then for any positive integer a we have*

$$(8.15) \quad F_{\mu,a}(\mathbf{y}) \equiv 0, \text{ i.e., } \mathcal{D}_{w_0^{(r-1)}} F_{\mu,a}(\mathbf{y}) = 0.$$

Call this statement (that (8.15) holds for any dominant weight μ and positive integer a) \mathbf{F}_r .

The argument in the present section amounts to the following lemma.

Lemma 27. *If $N_{r-1,r-2}$ holds, then $N_{r,r-1}$ (for λ, \mathbf{x} as above) is equivalent to the statement*

$$\forall a \quad F_{\mu,a}(\mathbf{y}) \equiv 0, \text{ or, equivalently, } \forall a \quad \mathcal{D}_{w_0^{(r-1)}} F_{\mu,a}(\mathbf{y}) = 0.$$

□

Now to complete the induction step, it remains to prove Proposition 26, i.e. that $F_{\mu,a}(\mathbf{y})$ is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$. This is the content of section 8.3.

8.3. Proof of Proposition 26. We distinguish between the cases where a is divisible by n or not. The case when it is *not* is significantly easier to handle.

8.3.1. The non-divisible case. The goal is to prove that if $n \nmid a$, then $\mathcal{D}_{w_0^{(r-1)}} F_{\mu,\Gamma_{11}}(\mathbf{y}) = 0$.

Recall that by Claim 5, $n \nmid a$ implies $h^b(a) = 0$. By (8.9), this means that $\delta(a, \Gamma_{12}) = 0$ unless $a = \Gamma_{12}$, and $\delta(a, \Gamma_{12}) = -1$. Thus in this case we have

$$(8.16) \quad F_{\mu,a}(\mathbf{y}) = - \sum_{\substack{\Gamma=(\Gamma_{12},\dots,\Gamma_{1r}) \\ \Gamma \text{ } \mu\text{-admissible} \\ \Gamma_{12}=a \\ \Gamma_{1,r} \neq 0}} G_1^{(\mu)}(\Gamma) \cdot \mathbf{y}^{\text{wt}(\mu)(\Gamma)} \cdot x_r^a.$$

We will show that each term in the summation is either itself zero, or is annihilated by a Demazure-Lusztig operator corresponding to a simple reflection. Fix a term $\Gamma = (\Gamma_{12}, \dots, \Gamma_{1r})$, and take $\Gamma_{1,r+1} := 0$ and $\Gamma_{11} = a$. Then $\Gamma_{11} = \Gamma_{12}$ and $\Gamma_{1r} > \Gamma_{1,r+1}$. Let j be the smallest index such that $\Gamma_{1j} > \Gamma_{1,j+1}$ ($2 \leq j \leq r$). In this case $a = \Gamma_{11} = \dots = \Gamma_{1,j-1} = \Gamma_{1,j}$, so $n \nmid \Gamma_{1,j}$ and hence $h^b(\Gamma_{1,j}) = 0$. By (6.10), we have

$$g_{1j}^{(\mu)}(\Gamma) = \begin{cases} h^b(\Gamma_{1,j}) & \Gamma_{1,j+1} < \Gamma_{1,j} < \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1; \\ g^b(\Gamma_{1,j}) & \Gamma_{1,j+1} < \Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1. \end{cases}$$

This means that $G_1^{(\mu)}(\Gamma) = 0$ unless $\Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1$. We show that in the latter case \mathcal{D}_{r-j+1} annihilates the corresponding term. Observe that $\Gamma_{1,j-1} = \Gamma_{1,j} = \Gamma_{1,j+1} + \lambda_j - \lambda_{j+1} + 1$ implies that $\mathbf{y}^{\text{wt}(\mu)(\Gamma)} \cdot x_r^{\Gamma_{11}}$ has a factor of $x_{r-j+1}^{\lambda_j+1} x_{r-j+2}^{\lambda_j}$ (and no other factors of x_{r-j+1} or x_{r-j+2}). By Corollary 9, \mathcal{D}_{r-j+1} indeed kills this term. Since $1 \leq r-j+1 \leq r-1$, by Proposition 8 part (ii) $\mathcal{D}_{w_0^{(r-1)}}$ annihilates all nonzero terms. This completes the proof of the non-divisible case.

8.3.2. *The divisible case.* From now on we assume that a is divisible by \mathbf{n} . Since $\delta(a, \Gamma_{12})$ will appear repeatedly in the computations below, we introduce the following shorthand. By Claim 5, we have

$$(8.17) \quad \delta_a(\Gamma_{12}) = \delta(a, \Gamma_{12}) = \begin{cases} 1-v & \text{if } a < \Gamma_{12}; \\ -v & \text{if } a = \Gamma_{12}; \\ 0 & \text{if } a > \Gamma_{12}. \end{cases}$$

The goal is to prove $\mathbf{F}_{\mathbf{r}}$:

$$(8.18) \quad \mathcal{D}_{w_0^{(r-1)}} F_{\mu,a}(\mathbf{y}) = \mathcal{D}_{w_0^{(r-1)}} \sum_{\substack{\Gamma=(\Gamma_{12}, \dots, \Gamma_{1r}) \\ \Gamma \text{ } \mu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} \delta_a(\Gamma_{12}) \cdot G_1^{(\mu)}(\Gamma) \cdot \mathbf{y}^{\text{wt}(\mu)(\Gamma)} \cdot x_r^a = 0.$$

Lemma 27 has the following convenient implication. Since as part of the inductive hypothesis, we assume that both $N_{r-1,r-2}$ and $N_{r-2,r-3}$ are true, we have the statement $\mathbf{F}_{\mathbf{r}-1}$: $\mathcal{D}_{w_0^{(r-2)}} F_{\nu, \Gamma_{12}}(\mathbf{z}) = 0$ for any Γ_{12} . This is true even when $r = 2$, since in this case, $F_{\nu, \Gamma_{12}}(\mathbf{z})$ is itself zero.

The strategy to prove (8.18) is the following. Using the conventions introduced in Section 8.1, in particular (8.7) and (8.8), we will rewrite the sum defining $F_{\mu,a}(\mathbf{y})$ into smaller pieces according to Γ_0 . Then we write $F_{\mu,a}(\mathbf{y})$ as a difference of two pieces. One piece is annihilated by $\mathcal{D}_{w_0^{(r-2)}}$ as a consequence of $\mathbf{F}_{\mathbf{r}-1}$, the other is annihilated by \mathcal{D}_{r-1} . By Lemma 8, this implies that $F_{\mu,a}(\mathbf{y})$ is indeed annihilated by $\mathcal{D}_{w_0^{(r-1)}}$.

We start by breaking up the sum in $F_{\mu,a}(\mathbf{y})$ according to Γ_0 . By the conventions introduced above, we have

$$(8.19) \quad \begin{aligned} F_{\mu,a}(\mathbf{y}) &= \sum_{\substack{\Gamma=(\Gamma_{12}, \dots, \Gamma_{1r}) \\ \Gamma \text{ } \mu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} \delta_a(\Gamma_{12}) \cdot g_{12}^{(\mu)}(\Gamma) \cdot G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\text{wt}(\nu)(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2 - \Gamma_{12} + a} \\ &= \sum_{\substack{\Gamma_0=(\Gamma_{13}, \dots, \Gamma_{1r}) \\ \Gamma_0 \text{ } \nu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\text{wt}(\nu)(\Gamma_0)} \cdot \sum_{\Gamma_{12}=\Gamma_{13}}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot g_{12}^{(\mu)}(\Gamma) \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2 - \Gamma_{12} + a} \end{aligned}$$

Recall that $\Gamma_0 = (\Gamma_{13}, \dots, \Gamma_{1r})$, where $\Gamma_{13} \geq \Gamma_{1r} \geq 1$, and if $\Gamma_{12} < \Gamma_{13}$ then $g_{12}^{(\mu)}(\Gamma) = 0$. Hence we may change the lower bound of the summation over Γ_{12} to 1. (If $r = 2$, then there is no change at all.)

$$(8.20) \quad F_{\mu,a}(\mathbf{y}) = \sum_{\substack{\Gamma_0=(\Gamma_{13}, \dots, \Gamma_{1r}) \\ \Gamma_0 \text{ } \nu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\text{wt}(\nu)(\Gamma_0)} \cdot \sum_{\Gamma_{12}=1}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot g_{12}^{(\mu)}(\Gamma) \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2 - \Gamma_{12} + a}$$

Next we derive from $\mathbf{F}_{\mathbf{r}-1}$ a term annihilated by $\mathcal{D}_{w_0^{(r-2)}}$. Recall that $\mathbf{F}_{\mathbf{r}-1}$ implies that $\mathcal{D}_{w_0^{(r-2)}} F_{\nu, \Gamma_{12}}(\mathbf{z}) = 0$ holds for any $0 < \Gamma_{12}$. The operator $\mathcal{D}_{w_0^{(r-2)}}$ is linear and commutes

with multiplication by x_r . This implies that for any positive integer a , we have

$$(8.21) \quad \mathcal{D}_{w_0^{(r-2)}} \sum_{0 < \Gamma_{12}} \delta_a(\Gamma_{12}) \cdot x_r^{\lambda_2 - \Gamma_{12} + a} \cdot F_{\nu, \Gamma_{12}}(\mathbf{z}) = 0.$$

We take a closer look at this polynomial annihilated in (8.21). It is

$$(8.22) \quad \sum_{0 < \Gamma_{12}} \delta_a(\Gamma_{12}) \cdot x_r^{\lambda_2 - \Gamma_{12} + a} \cdot \sum_{\substack{\Gamma_0 = (\Gamma_{13}, \dots, \Gamma_{1r}) \\ \Gamma_0 \text{ } \nu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} \delta(\Gamma_{12}, \Gamma_{13}) \cdot G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\text{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{12}}.$$

We may change the order of summation to get

$$(8.23) \quad \sum_{\substack{\Gamma_0 = (\Gamma_{13}, \dots, \Gamma_{1r}) \\ \Gamma_0 \text{ } \nu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\text{wt}^{(\nu)}(\Gamma_0)} \cdot \sum_{0 < \Gamma_{12}} \delta_a(\Gamma_{12}) \cdot \delta(\Gamma_{12}, \Gamma_{13}) \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2 - \Gamma_{12} + a}.$$

Since $\delta(\Gamma_{12}, \Gamma_{13}) = 0$ when $\Gamma_{12} > \Gamma_{13}$, the sum is unchanged if we put the upper bound $\Gamma_{13} + \lambda_2 - \lambda_3 + 1$ on the second summation:

$$(8.24) \quad \sum_{\substack{\Gamma_0 = (\Gamma_{13}, \dots, \Gamma_{1r}) \\ \Gamma_0 \text{ } \nu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\text{wt}^{(\nu)}(\Gamma_0)} \cdot \sum_{\Gamma_{12}=1}^{\Gamma_{13} + \lambda_2 - \lambda_3 + 1} \delta_a(\Gamma_{12}) \cdot \delta(\Gamma_{12}, \Gamma_{13}) \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2 - \Gamma_{12} + a}.$$

Comparing (8.24) to (8.20), we see that their shape is very similar. It suffices to show that their sum is annihilated by \mathcal{D}_{r-1} . We write the sum explicitly as follows.

$$(8.25) \quad \begin{aligned} F_{\mu,a}(\mathbf{y}) + \sum_{0 < \Gamma_{12}} \delta_a(\Gamma_{12}) \cdot x_r^{\lambda_2 - \Gamma_{12} + a} \cdot F_{\nu, \Gamma_{12}}(\mathbf{z}) &= \sum_{\substack{\Gamma_0 = (\Gamma_{13}, \dots, \Gamma_{1r}) \\ \Gamma_0 \text{ } \nu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\text{wt}^{(\nu)}(\Gamma_0)} \cdot \\ &\cdot \sum_{\Gamma_{12}=1}^{\Gamma_{13} + \lambda_2 - \lambda_3 + 1} \delta_a(\Gamma_{12}) \cdot (g_{12}^{(\mu)}(\Gamma) + \delta(\Gamma_{12}, \Gamma_{13})) \cdot x_{r-1}^{\Gamma_{12}} \cdot x_r^{\lambda_2 - \Gamma_{12} + a} \\ &= \sum_{\substack{\Gamma_0 = (\Gamma_{13}, \dots, \Gamma_{1r}) \\ \Gamma_0 \text{ } \nu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\text{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{13} - \lambda_3} \cdot \\ &\cdot \sum_{\Gamma_{12}=1}^{\Gamma_{13} + \lambda_2 - \lambda_3 + 1} \delta_a(\Gamma_{12}) \cdot (g_{12}^{(\mu)}(\Gamma) + \delta(\Gamma_{12}, \Gamma_{13})) \cdot x_{r-1}^{\lambda_3 - \Gamma_{13} + \Gamma_{12}} \cdot x_r^{\lambda_2 - \Gamma_{12} + a} \end{aligned}$$

Note that the exponent of x_{r-1} in $\mathbf{z}^{\text{wt}^{(\nu)}(\Gamma_0)}$ is $\lambda_3 - \Gamma_{13}$. This means that $\mathbf{z}^{\text{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{13} - \lambda_3}$ contains no factors of x_{r-1} or x_r . In particular, multiplication by $\mathbf{z}^{\text{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{13} - \lambda_3}$ commutes with \mathcal{D}_{r-1} . This implies that the sum in (8.25) is annihilated by \mathcal{D}_{r-1} if and only if the part corresponding to a fixed Γ_0 is.

Consider

$$(8.26) \quad h^{(\mu, \Gamma_0)}(\Gamma_{12}) := g_{12}^{(\mu)}(\Gamma) + \delta(\Gamma_{12}, \Gamma_{13}) = \begin{cases} h^b(\Gamma_{12}) & \text{if } \Gamma_{12} < \Gamma_{13} + \lambda_2 - \lambda_3 + 1; \\ g^b(\Gamma_{12}) & \text{if } \Gamma_{12} = \Gamma_{13} + \lambda_2 - \lambda_3 + 1; \\ 0 & \text{if } \Gamma_{12} > \Gamma_{13} + \lambda_2 - \lambda_3 + 1. \end{cases}$$

Let f_{a,Γ_0} be the polynomial that corresponds to a single Γ_0 in (8.25):

$$(8.27) \quad f_{a,\Gamma_0}(x_{r-1}, x_r) := \sum_{\Gamma_{12}=1}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot h^{(\mu,\Gamma_0)}(\Gamma_{12}) \cdot x_{r-1}^{\lambda_3+\Gamma_{12}-\Gamma_{13}} x_r^{\lambda_2+a-\Gamma_{12}}.$$

Lemma 28.

$$(8.28) \quad \mathcal{D}_{r-1} f_{a,\Gamma_0} = 0.$$

The proof of this lemma is a rank one computation using the definition of the group action. For completeness, it is included in Appendix A.

To summarize, we have

$$(8.29) \quad \begin{aligned} F_{\mu,a}(\mathbf{y}) = & \sum_{\substack{\Gamma_0=(\Gamma_{13},\dots,\Gamma_{1r}) \\ \Gamma_0 \text{ } \nu\text{-admissible} \\ \Gamma_{1,r} \neq 0}} G_1^{(\nu)}(\Gamma_0) \cdot \mathbf{z}^{\text{wt}^{(\nu)}(\Gamma_0)} \cdot x_{r-1}^{\Gamma_{13}-\lambda_3} \cdot f_{a,\Gamma_0}(x_{r-1}, x_r) \\ & - \sum_{0 < \Gamma_{12}} \delta_a(\Gamma_{12}) \cdot x_r^{\lambda_2-\Gamma_{12}+a} \cdot F_{\nu,\Gamma_{12}}(\mathbf{z}) \end{aligned}$$

The first term here is annihilated by \mathcal{D}_{r-1} ; the second is annihilated by $\mathcal{D}_{w_0^{(r-2)}}$. Hence $F_{\mu,a}(\mathbf{y})$ is annihilated by $\mathcal{D}_{w_0^{(r-1)}}$. This completes the proof of the statement \mathbf{F}_r (in the divisible case), and hence the proof of Proposition 24.

9. WHITTAKER FUNCTIONS

We mentioned in the Introduction that Theorem 2 establishes a combinatorial link between metaplectic analogues of the Casselman-Shalika formula. Furthermore, the more general Theorem 1 gives a crystal description of certain Iwahori-Whittaker functions. In this section, we make these statements more explicit.

We recall results of [CGP14] to compare the “Demazure-Lusztig side” of Theorem 2 to constructions in [McN16] and more specifically in type A to [CO13] (section 9.1). We relate the “crystal side” of Theorem 2 to Whittaker functions via comparison to [McN11] (section 9.2). In section 9.3 we compare the Demazure-Lusztig expression from Theorem 1 to constructions of Iwahori-Whittaker functions. We recall a relevant statement in three different contexts. For finite dimensional and affine Kac-Moody groups in the nonmetaplectic setting, through comparison with [BBL14] and [Pat14] respectively, and in the metaplectic setting by recalling results of [PP15].

9.1. Metaplectic Whittaker functions via Demazure-Lusztig operators. Formulae about Demazure operators of the long word, in particular Theorem 6 [CGP14, Theorem 3.] and Theorem 7 [CGP14, Theorem 4.] allow us to interpret the left-hand side of Theorem 1 as the value of a Whittaker function \mathcal{W} , constructed as a sum over the Weyl group in terms of the Chinta-Gunnells action. Such a construction can be found in [CO13] for type A , and in [McN16] in greater generality. The connection with results of [McN11] are made explicit in [CGP14, Section 6]. We recall that result here, and give the translation to results of [CO13] in the type A case.

9.1.1. *Metaplectic Whittaker functions.* We only sketch the definition of the Whittaker function \mathcal{W} and refer the reader to [CGP14, Section 6] for details and precise conditions in our notation. Let F be a non-archimedean local field containing the $2n$ -th roots of unity, ϖ the uniformizer of its ring of integers \mathcal{O} . Let G be a split, connected reductive group over F that arises as a special fibre over a group scheme \mathbf{G} defined over \mathbb{Z} . Let $K = \mathbf{G}(\mathcal{O})$ be the maximal compact subgroup of G , T a maximal split torus, B a Borel containing T , B^- its opposite, U the unipotent radical of B and U^- of the opposite Borel B^- . If Λ is the group of cocharacters of T then we may define a sublattice Λ_0 of Λ as in (3.6) (for the definition in general type, see [CGP14, Section 2, (3)]). Let \tilde{G} be an n -fold metaplectic cover of G . This in particular means that there is a short exact sequence

$$(9.1) \quad 1 \rightarrow \mu_n \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

where μ_n is the group of n -th roots of unity. We think of μ_n as being identified with a subgroup of \mathbb{C}^* and let \tilde{T} , the metaplectic torus (respectively, \tilde{B}) be the preimage of T (respectively, of B) in \tilde{G} . We shall give \mathcal{W} as a complex-valued Whittaker function corresponding to an unramified principal series representation of \tilde{G} . Let χ be a character of Λ_0 and $\psi : U^- \rightarrow \mathbb{C}$ an unramified character. Then χ determines an extension $\tilde{\chi}$ to \tilde{T} as well as a representation $\iota(\chi)$ of \tilde{T} (induced from its centralizer). In turn, $\iota(\chi)$ determines an unramified principal series representation of \tilde{G} . Let ϕ_K be a spherical vector; then there is a $\xi_{\tilde{\chi}} \in (\iota(\chi))^*$ a complex valued linear functional corresponding to χ and ϕ_K . From these we arrive at the Whittaker function

$$(9.2) \quad \mathcal{W} = \mathcal{W}_{\tilde{\chi}} : g \mapsto \xi_{\tilde{\chi}} \left(\int_{U^-} \phi_K(ug) \psi(u) du \right).$$

9.1.2. *Comparison.* It is a consequence of the construction (see [CGP14, Section 6]) that \mathcal{W} satisfies

$$\mathcal{W}(\zeta u g k) = \zeta \phi(u) \mathcal{W}(g), \zeta \in \mu_n, u \in U, g \in \tilde{G}, k \in K.$$

This fact and the Iwasawa decomposition $G = UTK$ together imply that it suffices to compute \mathcal{W} on \tilde{T} . The identification $\tilde{\chi}(\varpi^\lambda) = \mathbf{x}^\lambda$ interprets the action of the Weyl group on Λ as W acting on $\tilde{\chi}$. Setting $v = t^n = q^{-1}$ where q is the order of the residue field $\mathcal{O}/\varpi\mathcal{O}$, it makes sense to talk about a value $(\delta^{-1/2} \mathcal{W}_{\tilde{\chi}})(\varpi^\lambda)$ in terms of the expressions produced by metaplectic Demazure and Demazure-Lusztig operators acting on monomials. (Here δ is the modular quasicharacter of \tilde{B} .) In particular, by [McN16, Theorem 15.2], one has

Theorem 29. [CGP14, Theorem 16.] For a dominant coweight λ

$$(9.3) \quad \begin{aligned} (\delta^{-1/2} \mathcal{W}_{\tilde{\chi}})(\varpi^\lambda) &= \prod_{\alpha \in \Phi^+} (1 - q^{-1} x^{m(\alpha)\alpha}) \mathcal{D}_{w_0}(\mathbf{x}^{w_0\lambda}) \\ &= \left(\sum_{w \in W} \mathcal{T}_w \right) (\mathbf{x}^{w_0\lambda}) \end{aligned}$$

Note that the equality of the two lines on the right-hand side is a consequence of [CGP14, Theorem 4.] (i.e. Theorem 7). Theorem 9.3 is valid in any type.

We finish this comparison by arriving at the same statement in type A using results of [CO13]. As in [CO13, Section 9] define

$$(9.4) \quad j(w, x) = \frac{\prod_{\alpha \in \Phi^+} (1 - x^{m(\alpha)\alpha})}{\prod_{\alpha \in w(\Phi^+)} (1 - x^{m(\alpha)\alpha})}.$$

We have the following formula of Chinta-Offen:

Theorem 30. *[CO13, Theorem 4] Let λ be a dominant coweight. Then*

$$(\delta^{-1/2} \mathcal{W}_{\tilde{\chi}})(\varpi^\lambda) = c_{w_0}(x) \cdot \sum_{w \in W} j(w, x) \cdot w(x^{w_0 \lambda})$$

where w acts on x^λ as in Definition 1.

We may rewrite $j(w, x)$ in a more familiar form. A simple computations shows that for any $w \in W$ we have

$$(9.5) \quad j(w, x) = \text{sgn}(w) \cdot \prod_{\alpha \in \Phi(w^{-1})} x^{m(\alpha)\alpha}.$$

Combining Theorem 30 with [CGP14, Theorem 3.] (i.e. Theorem 6) we arrive at the type A special case of Theorem 29 once again.

9.2. Construction via crystals. McNamara [McN11, Section 8] expresses the value of a Whittaker function I_λ as a sum over a the highest weight crystal $\mathcal{C}_{\lambda+\rho}$. Here λ is a dominant weight and $I_{-w_0\lambda}$ is the same as $(\delta^{-1/2} \mathcal{W}_{\tilde{\chi}})(\varpi^\lambda)$ above, up to a relatively trivial constant factor. (It follows from our argument below that the value of this factor is in fact one.) The crystal $\mathcal{C}_{\lambda+\rho}$ is parametrized in [McN11] in terms of Lusztig data; to compare results we sketch the translation of McNamara's result into the notation of Gelfand-Tsetlin arrays.

9.2.1. Lusztig data and McNamara's result. We recall notation from [McN11, Section 8.]. Note first that the long word chosen in *loc. cit* agrees with our choice of w_0 from (4.10). Let $\lambda = (\lambda_1, \dots, \lambda_{r+1})$ and let us use the notation for a root system of type A_r as before (see section 3.1). In particular, recall that we have $\Phi^+ = \{\alpha_{i,j} = e_i - e_j \mid 1 \leq i < j \leq r+1\}$.

Proposition 31. *[McN11, Proposition 8.3] Elements of $\mathcal{C}_{\lambda+\rho}$ are in bijection with tuples $\mathbf{m} = (m_{ij})_{1 \leq i < j \leq r+1}$ where $0 \leq m_{ij}$ integers and*

$$(9.6) \quad \sum_{k=j}^{r+1} m_{i,k} \leq \lambda_i - \lambda_{i+1} + 1 + \sum_{k=j}^r m_{i+1,k+1}.$$

We write $\mathbf{m} \in \mathcal{C}_{\lambda+\rho}$ for tuples as above. For an $\alpha = \alpha_{i,j} \in \Phi^+$ we say $m_\alpha := m_{i,j}$ is *circled* if $m_{i,j} = 0$, and *boxed* if equality holds in (9.6). Furthermore, define

$$(9.7) \quad r_{i,j} = r_\alpha = \sum_{k \leq i} m_{k,j}.$$

Now we may use the functions h^\flat and g^\flat defined in Section 3.3.2 to define a coefficient corresponding to $\mathbf{m} \in \mathcal{C}_{\lambda+\rho}$. Let

$$(9.8) \quad w(\mathbf{m}, \alpha) = \begin{cases} 1 & \text{if } m_\alpha \text{ is circled, but not boxed,} \\ h^\flat(r_\alpha) & \text{if } m_\alpha \text{ is not circled and not boxed,} \\ g^\flat(r_\alpha) & \text{if } m_\alpha \text{ is boxed, but not circled,} \\ 0 & \text{if } m_\alpha \text{ is both circled and boxed} \end{cases}$$

It is straightforward to check that (9.8) defines the same weight as [McN11, (8.2)] by comparing section 3.3.2 with [McN11, (2.1)] and [BBF06, Lemma 2.5].

With the notation as above, one has

Theorem 32. [McN16, Theorem 8.6] *The value of the integral I_λ which calculates the metaplectic Whittaker function is zero unless λ is dominant; and for dominant λ it is given by*

$$(9.9) \quad I_\lambda = \sum_{\mathbf{m} \in \mathcal{C}_{\lambda+\rho}} \prod_{\alpha \in \Phi^+} w(\mathbf{m}, \alpha) \cdot x^{m_\alpha \alpha}.$$

As before, we write \mathbf{x} such that $\chi(\varpi^\lambda) = \mathbf{x}^\lambda$ for the unramified χ used to define the principal series representation.

9.2.2. Translation into Gelfand-Tsetlin language. It is convenient to compare the Lusztig data of $\mathcal{C}_{-w_0\lambda+\rho}$ to the Gelfand-Tsetlin arrays for $\mathcal{C}_{\lambda+\rho}$. For $\mathcal{C}_{-w_0\lambda+\rho}$, the condition (9.6) is replaced by $\mathbf{m} \in \mathcal{C}_{-w_0\lambda+\rho}$ if and only if $m_\alpha \geq 0$ and

$$(9.10) \quad \sum_{k=j}^{r+1} m_{i,k} \leq \lambda_{r+1-i} - \lambda_{r+2-i} + 1 + \sum_{k=j}^r m_{i+1,k+1};$$

and $m_{i,j}$ is boxed if (9.10) is satisfied with an equality.

Consider the following bijection between $\mathbf{m} \in \mathcal{C}_{-w_0\lambda+\rho}$ and $\Gamma(\mathfrak{T}_v)$ for $v \in \mathcal{C}_{\lambda+\rho}$:

$$(9.11) \quad \mathbf{m} \mapsto \Gamma(\mathfrak{T}) \quad \text{if} \quad \Gamma_{h,k} = r_{r+1-k, r+2-h}.$$

Note that (h, k) satisfy $1 \leq h \leq k \leq r$ if and only if $i = r+1-k$ and $j = r+2-h$ satisfy $1 \leq i < j \leq r+1$. Further, $m_{i,j} = r_{i,j} - r_{i-1,j}$. The bijection may be expressed in terms of the corresponding Γ -array as

$$(9.12) \quad m_{i,j} = \Gamma_{h,k} - \Gamma_{h,k+1}.$$

Thus $m_{i,j}$ is circled if and only if $\Gamma_{h,k} = \Gamma_{h,k+1}$, i.e. $\Gamma_{h,k}$ is circled by Definition 3. Similarly, one may check that $m_{i,j}$ is boxed if and only if $\sum_{t=j}^{r+1} m_{i,t} = a_{h-1,k-1} - a_{h,k}$ i.e. if and only if $\Gamma_{h,k}$ is boxed. Comparing the definition of $w(\mathbf{m}, \alpha)$ in (9.8) with the definition of the Gelfand-Tsetlin coefficients in 4, we find that for any $\mathbf{m} \in \mathcal{C}_{-w_0\lambda+\rho}$ and corresponding $v \in \mathcal{C}_{\lambda+\rho}$, we have

$$(9.13) \quad \prod_{\alpha \in \Phi^+} w(\mathbf{m}, \alpha) = G^{(n, \lambda+\rho)}(v).$$

An other direct computation shows that

$$(9.14) \quad \text{wt}(v) - w_0(\lambda + \rho) = \sum_{\alpha \in \Phi^+} m_\alpha \alpha.$$

Then Theorem 32, (9.13) and (9.14) immediately yields the following.

Theorem 33. *The value of the integral $I_{-w_0\lambda}$ which calculates the metaplectic Whittaker function is zero unless λ is dominant; and for dominant λ it is given by*

$$(9.15) \quad I_{-w_0\lambda} = \mathbf{x}^{-w_0(\lambda+\rho)} \cdot \sum_{v \in \mathcal{C}_{\lambda+\rho}} G^{(n, \lambda+\rho)}(v) \cdot \mathbf{x}^{\text{wt}(v)}.$$

The content of Theorem 33 identifies the “crystal side” of Theorem 2 as a metaplectic Whittaker function.

9.3. Constructions of Iwahori-Whittaker functions. The operators \mathcal{T}_u lend themselves to the study of Whittaker functions not only through expressions of the long word, as seen in section 9.1 but also through their relationship to Iwahori-Whittaker functions. This was mentioned in 1.3.1; in this section, we elaborate on this connection by recalling results of [BBL14], [Pat14] and [PP15]. These sources express Iwahori-Whittaker functions in terms reminiscent of the left-hand side of Theorem 1:

$$(9.16) \quad \sum_{u \leq w} \mathcal{T}_u$$

in the nonmetaplectic finite-dimensional, loop group, and metaplectic finite-dimensional setting, respectively.

9.3.1. The Whittaker functional and Iwahori fixed vectors. Brubaker, Bump and Licata [BBL14] consider the values of a Whittaker functional on Casselman's basis of functions in a principal series representation fixed by the Iwahori subgroup. They work with the classical, nonmetaplectic Demazure-Lusztig operators \mathfrak{T}_w ($w \in W$). (Their definition [BBL14, (2), (3)] essentially agrees with our Definition 3.19 when $n = 1$.)

We recall some additional notation; see [BBL14] for the precise definitions of the objects involved. Let \hat{T} be a split maximal torus of the Langlands dual $\hat{\mathbf{G}}$; as explained in [BBL14, Section 2], $\mathbf{z} \in \hat{T}(\mathbb{C})$ corresponds to an unramified character $\tau_{\mathbf{z}}$ of T . An element $\lambda \in \Lambda$ corresponds to a coset $T(F)/T(\mathcal{O})$; let a_{λ} be a coset representative. Consider the principal series representation $\pi = \text{Ind}_B^G(\tau)$. Let $\Omega_{\tau} : \text{Ind}_B^G(\tau) \rightarrow \mathbb{C}$ denote the Whittaker functional. Let J be the Iwahori subgroup (i.e. the preimage of $B^-(\mathcal{O}/\varpi\mathcal{O})$ in K), the space $\text{Ind}_B^G(\tau)^J$ of Iwahori fixed vectors has a standard basis $\{\Phi_w^{\mathbf{z}}\}_{w \in W}$, whose elements are supported on Iwahori double-cosets.

The Iwahori Whittaker functions $\mathcal{W}_{\lambda,y}(\mathbf{z})$ are (a convenient normalization of) the values of a Whittaker functional on standard basis elements:

$$(9.17) \quad \mathcal{W}_{\lambda,w}(\mathbf{z}) = \delta^{-1/2}(a_{\lambda}) \Omega_{\mathbf{z}^{-1}}(\pi(a_{-\lambda}) \Phi_w^{\mathbf{z}^{-1}}).$$

We may also consider [BBL14, Section 5] the modification $\widetilde{\mathcal{W}}_{\lambda,w}(\mathbf{z}) = \sum_{y \leq w} \mathcal{W}_{\lambda,y}(\mathbf{z})$. The connection between Iwahori Whittaker functions and Demazure-Lusztig operators is expressed by the following theorem.

Theorem 34. [BBL14, Theorem 1.] *For any dominant weight λ , we have $\mathcal{W}_{\lambda,1}(\mathbf{z}) = \mathbf{z}^{\lambda}$. Furthermore, if $w \in W$ and σ_i is a simple reflection such that $\sigma_i w > w$ by the Bruhat order, then*

$$\mathcal{W}_{\lambda,\sigma_i w} = \mathfrak{T}_i \mathcal{W}_{\lambda,w}(\mathbf{z}).$$

The following straightforward corollary illustrates the relevance of operators (9.16).

Corollary 35. *For any dominant weight λ and $w \in W$, we have*

$$\widetilde{\mathcal{W}}_{\lambda,w}(\mathbf{z}) = \left(\sum_{y \leq w} \mathfrak{T}_y \right) \mathbf{z}^{\lambda}.$$

□

9.3.2. Iwahori-Whittaker sums on loop groups. In this section we shift our perspective slightly. We recall results of Patnaik [Pat14] that demonstrate the use of Demazure-Lusztig operators in the study of Whittaker functions in yet another setting: on p -adic points of an affine Kac-Moody group.

Let G be an affine Kac-Moody group over a non-archimedean field; ϖ, q, K, U, U^- as before. Let W now denote the (affine) Weyl group of G ; and Π_0 the basis of the corresponding finite root system. Let I and I^- denote the Iwahori subgroups. In [Pat14, Section 4] the Whittaker function \mathcal{W} is defined on G . Furthermore, determining \mathcal{W} is reduced to the computation of values $\mathcal{W}(\varpi^{\lambda^\vee})$, for any $\lambda^\vee \in \Lambda^\vee$ affine coweight. The main theorem [Pat14, Theorem 7.1], a generalization of the Casselman-Shalika formula for the computation of $\mathcal{W}(\varpi^{\lambda^\vee})$ is proved through the introduction of Iwahori-Whittaker sums $\mathcal{W}_{w,\lambda^\vee}$ and a recursion result on $\mathcal{W}_{w,\lambda^\vee}$ in terms of Demazure-Lusztig operators. The recursion result [Pat14, Proposition 5.5] is recalled in Proposition 36 below.

The definition [Pat14, (2.29)] of Demazure-Lusztig operators T_a ($a \in \Pi_0$) essentially agrees with Definition 3.19 when $n = 1$ and the root system is of type A . The Iwahori-Whittaker sums $\mathcal{W}_{w,\lambda^\vee}$ are defined [Pat14, Definition 4.5] by summing an unramified principal character ψ of U^- along fibres of the multiplication map $m_{w,\lambda^\vee} : UwI^- \times_{I^-} I^- \varpi^{\lambda^\vee} U^- \rightarrow G$:

$$(9.18) \quad \mathcal{W}_{w,\lambda^\vee} := \sum_{\mu^\vee} e^{\mu^\vee} q^{\langle \rho, \mu^\vee \rangle} \sum_{x \in m_{w,\lambda^\vee}^{-1}(\varpi^{\mu^\vee})} \psi(x)$$

(For details on how to interpret the unramified character ψ on elements of the fibre, see *loc.cit.*)

The Whittaker function may then be written as a sum of these Iwahori-Whittaker sums [Pat14, (4.21)]: $\mathcal{W}(\pi^{\lambda^\vee}) = \sum_{w \in W} \mathcal{W}_{w,\lambda^\vee}$. The following proposition phrases the recursion of the Iwahori-Whittaker sums in terms of Demazure-Lusztig operators.

Proposition 36. [Pat14, Proposition 5.5] *Fix $\lambda^\vee \in \Lambda^\vee$ and let $w, w' \in W$ be related by $w = \sigma_a w'$ with $a \in \Pi_0$ a simple root, and $\ell(w) = \ell(w') + 1$. Then the following identity holds in $\mathbb{C}_{\sigma_a}[\Lambda^\vee]$:*

$$(9.19) \quad \mathcal{W}_{w,\lambda^\vee} = T_a(\mathcal{W}_{w',\lambda^\vee}).$$

(Here the simple reflection in T_a acts on $\mathcal{W}_{w',\lambda^\vee}$ termwise; the ring $\mathbb{C}_{\sigma_a}[\Lambda^\vee]$ is an extension of $\mathbb{C}[\Lambda^\vee]$ containing $T_a(\mathcal{W}_{w',\lambda^\vee})$.)

This recursion is used in [Pat14, Section 7.2] to conclude that $\mathcal{W}_{w,\lambda^\vee} = q^{\rho, \lambda^\vee} T_w(e^{\lambda^\vee})$ [Pat14, (7.3)] and to compute the value of $\mathcal{W}(\varpi^{\lambda^\vee})$, proving the generalization of the Casselman-Shalika formula.

9.3.3. The metaplectic setting. Recent results of Manish Patnaik and the present author [PP15] indicate that Demazure-Lusztig operators can be used to express Iwahori-Whittaker functions directly in the metaplectic setting as well. In particular, the techniques of [Pat14] seen above are applicable in the finite dimensional metaplectic setting. The Iwahori-Whittaker functions $\mathcal{W}_{w,\lambda^\vee}$ can again be defined as a formal generating series using fibres of the map m_{w,λ^\vee} ; an argument similar to that in [Pat14] proves that the value of the metaplectic Whittaker function $\mathcal{W}(\pi^{\lambda^\vee})$ can be expressed as a sum of these: $\mathcal{W}(\pi^{\lambda^\vee}) = q^{-(2\rho, \lambda^\vee)} \sum_{w \in W} \mathcal{W}_{w,\lambda^\vee}$ [PP15, Section 5.2]. It turns out that then the $\mathcal{W}_{w,\lambda^\vee}$ satisfy a similar

recursion to the one in Proposition 36 above [PP15, Theorem 5.4], and, consequently, writing \mathcal{T}_w for the metaplectic Demazure-Lusztig operators, we have $\mathcal{W}_{w,\lambda^\vee} = q^{(2\rho, \lambda^\vee)} \mathcal{T}_w(e^{\lambda^\vee})$ [PP15, Corollary 5.4].

APPENDIX A. PROOF OF LEMMA 28

We prove Lemma 28:

$$(A.1) \quad \mathcal{D}_{r-1} f_{a,\Gamma_0}(x_{r-1}, x_r) = \mathcal{D}_{r-1} \sum_{\Gamma_{12}=1}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) h^{(\mu, \Gamma_0)}(\Gamma_{12}) \cdot x_{r-1}^{\lambda_3+\Gamma_{12}-\Gamma_{13}} x_r^{\lambda_2+a-\Gamma_{12}} = 0.$$

By Lemma 8, it suffices to show $x_r^n \cdot f_{a,\Gamma_0}$ is symmetric under the action of σ_{r-1} . Since $\delta_a(\Gamma_{12}) = 0$ if $a > \Gamma_{12}$, we have

$$(A.2) \quad x_r^n \cdot f_{a,\Gamma_0} = \sum_{\Gamma_{12}=a}^{\Gamma_{13}+\lambda_2-\lambda_3+1} \delta_a(\Gamma_{12}) \cdot h^{(\mu, \Gamma_0)}(\Gamma_{12}) \cdot x_{r-1}^{\lambda_3+\Gamma_{12}-\Gamma_{13}} x_r^{\lambda_2+a-\Gamma_{12}+n}.$$

The proof is a straightforward computation. The strategy is as follows. By (8.26), $h^{(\mu, \Gamma_0)}$ depends on the residue of Γ_{12} modulo n . We write

$$(A.3) \quad (\Gamma_{13} + \lambda_2 - \lambda_3 + 1) - a = nk + u, \quad 1 \leq u \leq n.$$

We define

$$(A.4) \quad P_{k,u}(\mathbf{x}) = \frac{x_r^n \cdot f_{a,\Gamma_0}}{(1-v) \cdot (x_{r-1} x_r)^{\lambda_3+a-\Gamma_{13}}}.$$

Since by (3.11) σ_{r-1} commutes with multiplication by $(x_{r-1} x_r)$, proving (A.1) is equivalent to showing that $\sigma_{r-1}(P_{k,u}(\mathbf{x})) = P_{k,u}(\mathbf{x})$. In what follows, we write σ for σ_{r-1} and α for α_{r-1} . To prove that $P_{k,u}(\mathbf{x})$ is invariant under σ , we show that

$$(A.5) \quad P_{k,u}(\mathbf{x}) = \frac{1-v \cdot \mathbf{x}^{-n\alpha}}{\mathbf{x}^{-n\alpha} - 1} \cdot x_r^{nk+u+n-1} + \frac{1-v \cdot \mathbf{x}^{n\alpha}}{\mathbf{x}^{n\alpha} - 1} \cdot \sigma(x_r^{nk+u+n-1}).$$

This is sufficient by Lemma 3.

We are now ready to start the computation. By (A.2), (A.3) and (A.4) we have

$$(A.6) \quad P_{k,u}(\mathbf{x}) = \frac{1}{1-v} \cdot x_r^{nk+u+n-1} \cdot \sum_{\Gamma_{12}=a}^{a+nk+u} \delta_a(\Gamma_{12}) \cdot h^{(\mu, \Gamma_0)}(\Gamma_{12}) \cdot x_{r-1}^{\Gamma_{12}-a} x_r^{-(\Gamma_{12}-a)}$$

Recall that since $n|a$, we have

$$\delta_a(\Gamma_{12}) = \begin{cases} -v & a = \Gamma_{12}; \\ 1-v & a < \Gamma_{12}; \end{cases}$$

$$h^{(\mu, \Gamma_0)}(\Gamma_{12}) = \begin{cases} h^b(\Gamma_{12}) = h^b(\Gamma_{12}-a) & \text{if } \Gamma_{12} < \Gamma_{13} + \lambda_2 - \lambda_3 + 1 = a + nk + u; \\ g^b(\Gamma_{12}) = g^b(\Gamma_{12}-a) & \text{if } \Gamma_{12} = \Gamma_{13} + \lambda_2 - \lambda_3 + 1 = a + nk + u. \end{cases}$$

Furthermore, by Claim 5 we have $h^b(\Gamma_{12}-a) = 0$ if $n \nmid \Gamma_{12}-a$. Hence the terms of $P_{k,u}(\mathbf{x})$ where $n \nmid \Gamma_{12}-a$ are zero. Furthermore, by Claim 5 we also have $h^b(\Gamma_{12}-a) = 1-v$ if

$n|\Gamma_{12} - a$ and $g^\flat(\Gamma_{12} - a) = v \cdot g_{\Gamma_{12}-a}$. Substituting into (A.6), we get

$$\begin{aligned}
 P_{k,u}(\mathbf{x}) &= x_r^{nk+u+n-1} \cdot \frac{1}{1-v} \cdot \sum_{\Gamma_{12}=a}^{a+nk+u} \delta_a(\Gamma_{12}) \cdot h^{(\mu, \Gamma_0)}(\Gamma_{12}) \cdot \mathbf{x}^{(\Gamma_{12}-a)\alpha} \\
 &= \frac{1}{1-v} \cdot x_r^{nk+u+n-1} \cdot \left((-v) \cdot (1-v) + \sum_{i=1}^k (1-v)^2 \cdot \mathbf{x}^{in\alpha} + (1-v) \cdot v \cdot g_u \cdot \mathbf{x}^{(nk+u)\alpha} \right) \\
 &= x_r^{nk+u+n-1} \cdot \left((-v) + (1-v) \cdot \frac{\mathbf{x}^{(nk+n)\alpha} - \mathbf{x}^{n\alpha}}{\mathbf{x}^{n\alpha} - 1} + v \cdot g_u \cdot \mathbf{x}^{(nk+u)\alpha} \right)
 \end{aligned}$$

To rewrite this in the form of (A.5), note that by the definition of the Chinta-Gunnells action in type A (3.11), we have

$$\begin{aligned}
 (A.8) \quad \frac{1-v \cdot \mathbf{x}^{n\alpha}}{\mathbf{x}^{n\alpha} - 1} \cdot \sigma(x_r^{nk+u+n-1}) &= \\
 &= \frac{x_r^{nk+u+n-1}}{\mathbf{x}^{n\alpha} - 1} \cdot \left(\mathbf{x}^{-r_n(nk+u+n-1)\alpha} \cdot (1-v) - v \cdot g_{nk+u+n} \cdot \mathbf{x}^{(1-n)\alpha} \cdot (1 - \mathbf{x}^{n\alpha}) \right) \\
 &= x_r^{nk+u+n-1} \cdot \mathbf{x}^{(nk+u+n-1)\alpha} \cdot \left(\frac{\mathbf{x}^{(1-u)\alpha} \cdot (1-v)}{\mathbf{x}^{n\alpha} - 1} + v \cdot g_u \cdot \mathbf{x}^{(1-n)\alpha} \right) \\
 &= x_r^{nk+u+n-1} \cdot \left(\frac{\mathbf{x}^{(nk+n)\alpha} \cdot (1-v)}{\mathbf{x}^{n\alpha} - 1} + v \cdot g_u \cdot \mathbf{x}^{(nk+u)\alpha} \right)
 \end{aligned}$$

Comparing (A.8) to (A.7) we see that

$$\begin{aligned}
 (A.9) \quad P_{k,u}(\mathbf{x}) - \frac{1-v \cdot \mathbf{x}^{n\alpha}}{\mathbf{x}^{n\alpha} - 1} \cdot \sigma(x_r^{nk+u+n-1}) &= x_r^{nk+u+n-1} \cdot \left((-v) + (1-v) \cdot \frac{-\mathbf{x}^{n\alpha}}{\mathbf{x}^{n\alpha} - 1} \right) \\
 &= x_r^{nk+u+n-1} \cdot \frac{1-v \cdot \mathbf{x}^{-n\alpha}}{\mathbf{x}^{-n\alpha} - 1}.
 \end{aligned}$$

This completes the proof of (A.5), and thus of Lemma 28.

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